Section 4.8. Independent Random Variables

Note. Recall from Definition 4.3.1 that two events A and B are *independent* if $P(A \cap B) = P(A)P(B)$. In this section we define independent random variables and relate this to properties of the distribution functions and the densities.

Definition 4.8.1. Let $X_1, X_2, \ldots, X_n$ be random variables on $(\Omega, \mathcal{F}, P)$. Then $X_1, X_2, \ldots, X_n$ are *independent* if for all sets $B_1, B_2, \ldots, B_n \in \mathcal{B}(\mathbb{R})$ we have

$$P(\{X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n\})$$

$$= P(\{X_1 \in B_1\})P(\{X_2 \in B_2\}) \cdots P(\{x_n \in B_n\}).$$

We similarly define independent extended random variable by replacing $\mathcal{B}(\mathbb{R})$ with $\mathcal{B}(\overline{\mathbb{R}}) = \mathcal{B}(\mathbb{R} \cup \{-\infty, \infty\})$.

Note. If $X_1, X_2, \ldots, X_n$ are independent random variables, then $X_1, X_2, \ldots, X_k$, for $k < n$, are also random events. To see this, let $B_1, B_2, \ldots, B_k \in \mathcal{B}(\mathbb{R})$ and take $B_{k+1} = B_{k+2} = \cdots = B_n = \mathbb{R}$. Then

$$\begin{aligned} & P(\{X_1 \in B_1, X_2 \in B_2, \ldots, X_k \in B_k\}) \\ = \;& P(\{X_1 \in B_1, X_2 \in B_2, \ldots, X_k \in B_k, X_{k+1} \in \mathbb{R}, X_{k+2} \in \mathbb{R}, \ldots, X_n \in \mathbb{R}\}) \\ = \;& P(\{X_1 \in B_1\})P(\{X_2 \in B_2\}) \cdots P(\{X_k \in B_k\}) \cdot 1 \cdot 1 \cdots 1 \\ = \;& P(\{X_1 \in B_1\})P(\{X_2 \in B_2\}) \cdots P(\{X_k \in B_k\}). \end{aligned}$$

Notice that this behavior of random variables is different from the behavior of events, as illustrated in Note 4.3.A.

Definition. A *random object* on $(\Omega, \mathcal{F}, P)$ is a map $X_i : (\Omega, \mathcal{F}) \to (\Omega_i, \mathcal{F}_i)$ where ω_i is a set and $\mathcal{F}_i$ is a σ-field of subsets of Ω_i. Random objects $X_1, X_2, \ldots, X_n$ are independent if for all $B_1 \in \mathcal{F}_1$, $B_2 \in \mathcal{F}_2$, $\ldots$, $B_n \in \mathcal{F}_n$ we have

$$P(\{X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n\})$$

$$= P(\{X_1 \in B_1\})P(\{X_2 \in B_2\}) \cdots P(\{X_n \in B_n\}).$$

An arbitrary family of random objects X_i, where $i \in I$, is *independent* if $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ are independent for all finite sets $\{i_1, I_2, \ldots, i_n\}$ of distinct indices in I.

Note. We now explore the classification of independent random variables in terms of properties of the distribution function. First we need some results from Chapter 1 of the text.

Definition. Let Ω be a set. The collection of subsets $\mathcal{F}_0$ of set Ω is a *field* (or *algebra*) if it contians Ω and it is closed under finite unions and complements (and finite intersections, by De Morgan's Laws). Collection $\mathcal{C}$ of subsets on ω is *monotone* if (1) $\{A_m\}_{n=1}^\infty \subset \mathcal{C}$ with $A_1 \subset A_2 \subset \cdots$ implies $\lim_{n\to\infty} A_n = \cup_{n=1}^\infty \in \mathcal{C}$, and (2) if $\{B_n\}_{n=1}^\infty \subset \mathcal{C}$ with $B_1 \supset B_2 \supset \cdots$ implies $\lim_{n\to\infty} B_n = \cap_{n=1}^\infty B_n \in \mathcal{C}$. If $\mathcal{F}$ is a collection of subsets of Ω then the *minimal σ-field over $\mathcal{F}$*, denoted $\sigma(\mathcal{F})$, is the intersection of all σ-fields containing $\mathcal{F}$ (this is called the "the smallest σ-algebra that contains $\mathcal{F}$" of the σ-algebra generated by $\mathcal{F}$; see their Proposition 1.13).

Note. "Clearly" an arbitrary intersection of monotone collections is a monotone collection so we speak of the smallest monotone class containing a given collection of sets. We need the following in this section.

Theorem 1.3.9. The Monotone Class Theorem.

Let $\mathcal{F}_0$ be a field of subsets of Ω and $\mathcal{C}$ a class of subsets of Ω that is monotone. If $\mathcal{C} \supset \mathcal{F}_0$, then $\mathcal{C} \supset \sigma(\mathcal{F}_0)$, the minimal σ-filed over $\mathcal{F}_0$.

Theorem 4.8.3. Let $X_1, X_2, \ldots, X_n$ be random variables on $(\Omega, \mathcal{F}, P)$. Let F_i by the distribution function of X_i, $i = 1, 2, \ldots, n$ (so $F_i(x_i) = P(\{X_i \leq x_i\})$) and F the distribution function of $X = (X_1, X_2, \ldots, X_n)$ (that is, $F(x_1, x_2, \ldots, x_n) = P(\{X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n\})$). Then $X_1, X_2, \ldots, X_n$ are independent if and only if $F(x_1, x_2, \ldots, x_n) = F_1(x_1)F_2(x_2)\cdots F_n(x_n)$ for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

Note. The next result lets us classify independent random variables in terms of proerties of the density function.

Theorem 4.8.4. If $X = (X_1, X_2, \ldots, X_n)$ has a density function f, then each X_i has a density f_i. Furthermore, in this case $X_1, X_2, \ldots, X_n$ are independent if and only if $f(x_1, x_2, \ldots, x_n) = f_1(x_1)f_2(x_1)\cdots f_n(x_n)$ for all $(x_1, x_2, \ldots, x_n)$ except possibly for a Borel subset of $\mathbb{R}^n$ with Lebesgue measure zero.

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