Lemma 1.3.1. There exists only one set with no elements.

Proof. Suppose $A$ and $B$ are both sets with no elements. Then every element of $A$ is an element of $B$ (vacuously) and every element of $B$ is an element of $A$ (vacuously). Therefore, by the Axiom of Extensionality, $A = B$. □

Example/Theorem 1.3.3. If $P$ and $Q$ are sets then there is a set $R$ such that $x \in R$ if and only if $x \in P$ and $x \in Q$.

Proof. Let property $R(x, Q)$ be “$x \in Q$.” Then by the Axiom Schema of Comprehension, for any give set $A = P$ there is a set $B = R$ such that $x \in R$ if and only if $x \in P$ and $P(x, Q)$; that is, $x \in R$ if and only if $x \in P$ and $x \in Q$. □

Lemma 1.3.4. For every set $A$ and every property $P(x)$, there is only one set $B$ such that $x \in B$ if and only if $x \in A$ and $P(x)$.

Proof. Suppose $B'$ is another set such that $x \in B'$ if and only if $x \in A$ and $P(x)$. Then $x \in B$ if and only if $x \in B'$; that is, every element of $B$ is an element of $B'$ and every element of $B'$ is an element of $B$. So by the Axiom of Extensionality we have that $B' = B$. Hence, set $B$ is unique. □
Example/Theorem 1.3.13.

(a) \( \{ x \mid x \in P \text{ and } x \in Q \} \) exists.
(b) \( \{ a \mid x = a \text{ or } x = b \} \) exists.
(c) \( \{ x \mid x \notin x \} \) does not exist. (This is Russell’s Paradox.)

Proof. (a) Let \( P(x, P, Q) \) be the property “\( x \in P \) and \( x \in Q \).” Let \( A = P \). Then \( P(x, P, Q) \) implies \( x \in P \). So by Exercise 1.3.E,
\[
\{ x \in P \mid P(x, P, Q) \} = \{ x \in P \mid x \in P \text{ and } x \in Q \}
= \{ x \mid x \in P \text{ and } x \in Q \} \text{ exists. In fact, this is set } R \text{ of Example 1.3.3.}
\]

(b) Let \( A = \{ a, b \} \) which exists by the Axiom of Pair. So
\[
\{ x \mid x = a \text{ or } x = b \} = \{ a, b \} \text{ exists.}
\]

(c) Let \( P(x) \) be the property “\( x \notin x \).” Suppose such a set exists,
\( R = \{ x \mid x \notin x \} = \{ x \mid P(x) \} \). If \( R \in R \) then property \( P(R) \) holds (since \( R \) is in set \( R \)) and so \( R \notin R \), a contradiction. If \( R \notin R \) then property \( R(R) \) holds (by the definition of \( P \)) and so \( R \in R \), a contradiction.