Chapter 1. Sets

1.3. The Axioms—Proofs of Theorems
Table of contents

1. Lemma 1.3.1
2. Example/Theorem 1.3.3
3. Theorem 1.3.4
4. Example/Theorem 1.3.13
**Lemma 1.3.1.** There exists only one set with no elements.

**Proof.** Suppose $A$ and $B$ are both sets with no elements. Then every element of $A$ is an element of $B$ (vacuously) and every element of $B$ is an element of $A$ (vacuously). Therefore, by the Axiom of Extensionality, $A = B$. 

\[
\square
\]
Lemma 1.3.1. There exists only one set with no elements.

Proof. Suppose $A$ and $B$ are both sets with no elements. Then every element of $A$ is an element of $B$ (vacuously) and every element of $B$ is an element of $A$ (vacuously). Therefore, by the Axiom of Extensionality, $A = B$. 

\qed
Example/Theorem 1.3.3. If $P$ and $Q$ are sets then there is a set $R$ such that $x \in R$ if and only if $x \in P$ and $x \in Q$.

Proof. Let property $R(x, Q)$ be “$x \in Q$.” Then by the Axiom Schema of Comprehension, for any given set $A = P$ there is a set $B = R$ such that $x \in R$ if and only if $x \in P$ and $P(x, Q)$; that is, $x \in R$ if and only if $x \in P$ and $x \in Q$. □
Example/Theorem 1.3.3. If $P$ and $Q$ are sets then there is a set $R$ such that $x \in R$ if and only if $x \in P$ and $x \in Q$.

**Proof.** Let property $R(x, Q)$ be “$x \in Q$.” Then by the Axiom Schema of Comprehension, for any give set $A = P$ there is a set $B = R$ such that $x \in R$ if and only if $x \in P$ and $P(x, Q)$; that is, $x \in R$ if and only if $x \in P$ and $x \in Q$. 

\[\square\]
Theorem 1.3.4

Lemma 1.3.4. For every set $A$ and every property $P(x)$, there is only one set $B$ such that $x \in B$ if and only if $x \in A$ and $P(x)$.

Proof. Suppose $B'$ is another set such that $x \in B'$ if and only if $x \in A$ and $P(x)$. Then $x \in B$ if and only if $x \in B'$; that is, every element of $B$ is an element of $B'$ and every element of $B'$ is an element of $B$. So by the Axiom of Extensionality we have that $B' = B$. Hence, set $B$ is unique. \qed
Lemma 1.3.4. For every set $A$ and every property $P(x)$, there is only one set $B$ such that $x \in B$ if and only if $x \in A$ and $P(x)$.

Proof. Suppose $B'$ is another set such that $x \in B'$ if and only if $x \in A$ and $P(x)$. Then $x \in B$ if and only if $x \in B'$; that is, every element of $B$ is an element of $B'$ and every element of $B'$ is an element of $B$. So by the Axiom of Extensionality we have that $B' = B$. Hence, set $B$ is unique. \qed
Example/Theorem 1.3.13.

(a) \( \{ x \mid x \in P \text{ and } x \in Q \} \) exists.

(b) \( \{ a \mid x = a \text{ or } x = b \} \) exists.

(c) \( \{ x \mid x \not\in x \} \) does not exist. (This is Russell’s Paradox.)

Proof. (a) Let \( P(x, P, Q) \) be the property “\( x \in P \text{ and } x \in Q \).” Let \( A = P \). Then \( P(x, P, Q) \) implies \( x \in P \). So by Exercise 1.3.E, \[ \{ x \in P \mid P(x, P, Q) \} = \{ x \in P \mid x \in P \text{ and } x \in Q \} = \{ x \mid x \in P \text{ and } x \in Q \} \] exists. In fact, this is set \( R \) of Example 1.3.3.
Example/Theorem 1.3.13.

(a) \( \{x \mid x \in P \text{ and } x \in Q\} \) exists.
(b) \( \{a \mid x = a \text{ or } x = b\} \) exists.
(c) \( \{x \mid x \notin x\} \) does not exist. (This is Russell's Paradox.)

Proof. (a) Let \( P(x, P, Q) \) be the property “\( x \in P \text{ and } x \in Q \).” Let \( A = P \). Then \( P(x, P, Q) \) implies \( x \in P \). So by Exercise 1.3.E, 
\[
\{x \in P \mid P(x, P, Q)\} = \{x \in P \mid x \in P \text{ and } x \in Q\} = \{x \mid x \in P \text{ and } x \in Q\} \text{ exists. In fact, this is set } R \text{ of Example 1.3.3.}
\]

(b) Let \( A = \{a, b\} \) which exists by the Axiom of Pair. So 
\[
\{x \mid x = a \text{ or } x = b\} = \{a, b\} \text{ exists.}
\]
Example/Theorem 1.3.13.

(a) \( \{ x \mid x \in P \text{ and } x \in Q \} \) exists.
(b) \( \{ a \mid x = a \text{ or } x = b \} \) exists.
(c) \( \{ x \mid x \not\in x \} \) does not exist. (This is Russell’s Paradox.)

Proof. (a) Let \( P(x, P, Q) \) be the property “\( x \in P \text{ and } x \in Q \).” Let \( A = P \). Then \( P(x, P, Q) \) implies \( x \in P \). So by Exercise 1.3.E,

\[
\{ x \in P \mid P(x, P, Q) \} = \{ x \in P \mid x \in P \text{ and } x \in Q \}
\]

= \( \{ x \mid x \in P \text{ and } x \in Q \} \) exists. In fact, this is set \( R \) of Example 1.3.3.

(b) Let \( A = \{ a, b \} \) which exists by the Axiom of Pair. So

\[
\{ x \mid x = a \text{ or } x = b \} = \{ a, b \} \text{ exists.}
\]

(c) Let \( P(x) \) be the property “\( x \not\in x \).” Suppose such a set exists, \( R = \{ x \mid x \not\in x \} = \{ x \mid P(x) \} \). If \( R \in R \) then property \( P(R) \) holds (since \( R \) is in set \( R \)) and so \( R \not\in R \), a contradiction. If \( R \not\in R \) then property \( R(R) \) holds (by the definition of \( P \)) and so \( R \in R \), a contradiction.
Example/Theorem 1.3.13.

(a) \( \{x \mid x \in P \text{ and } x \in Q\} \) exists.

(b) \( \{a \mid x = a \text{ or } x = b\} \) exists.

(c) \( \{x \mid x \not\in x\} \) does not exist. (This is Russell’s Paradox.)

Proof. (a) Let \( P(x, P, Q) \) be the property “\( x \in P \) and \( x \in Q \).” Let \( A = P \). Then \( P(x, P, Q) \) implies \( x \in P \). So by Exercise 1.3.E, \( \{x \in P \mid P(x, P, Q)\} = \{x \in P \mid x \in P \text{ and } x \in Q\} \)
\( = \{x \mid x \in P \text{ and } x \in Q\} \) exists. In fact, this is set \( R \) of Example 1.3.3. □

(b) Let \( A = \{a, b\} \) which exists by the Axiom of Pair. So \( \{x \mid x = a \text{ or } x = b\} = \{a, b\} \) exists. □

(c) Let \( P(x) \) be the property “\( x \not\in x\).” Suppose such a set exists, \( R = \{x \mid x \not\in x\} = \{x \mid P(x)\} \). If \( R \in R \) then property \( P(R) \) holds (since \( R \) is in set \( R \)) and so \( R \not\in R \), a contradiction. If \( R \not\in R \) then property \( R(R) \) holds (by the definition of \( P \)) and so \( R \in R \), a contradiction. □