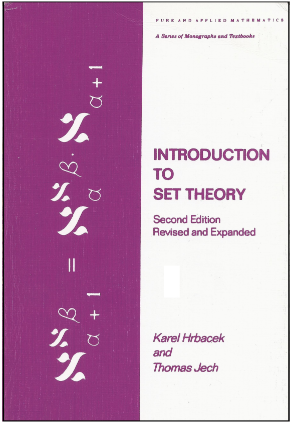
Introduction to Set Theory

Chapter 1. Sets

1.3. The Axioms—Proofs of Theorems



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Lemma 1.3.1

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Example/Theorem 1.3.3

Example/Theorem 1.3.3. If P and Q are sets then there is a set R such that $x \in R$ if and only if $x \in P$ and $x \in Q$.

Proof. Let property $\mathbf{R}(x, Q)$ be "$x \in Q$." Then by the Axiom Schema of Comprehension, for any give set $A = P$ there is a set $B = R$ such that $x \in R$ if and only if $x \in P$ and $\mathbf{P}(x, Q)$; that is, $x \in R$ if and only if $x \in P$ and $x \in Q$. □

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Theorem 1.3.4

Lemma 1.3.4. For every set A and every property $\mathbf{P}(x)$, there is only one set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

Proof. Suppose B' is another set such that $x \in B'$ if and only if $x \in a$ and $\mathbf{P}(x)$. Then $x \in B$ if and only if $x \in B'$; that is, every element of B is an element of B' and every element of B' is an element of B. So by the Axiom of Extensionality we have that $B' = B$. Hence, set B is unique. □

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Example/Theorem 1.3.13

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(a) $\{x \mid x \in P \text{ and } x \in Q\}$ exists.

(b) $\{a \mid x = a \text{ or } x = b\}$ exists.

(c) $\{x \mid x \notin x\}$ does not exist. (This is Russell's Paradox.)

Proof. **(a)** Let $\mathbf{P}(x, P, Q)$ be the property "$x \in P$ and $x \in Q$." Let $A = P$. Then $\mathbf{P}(x, P, Q)$ implies $x \in P$. So by Exercise 1.3.E, $\{x \in P \mid \mathbf{P}(x, P, Q)\} = \{x \in P \mid x \in P \text{ and } x \in Q\}$ $= \{x \mid x \in P \text{ and } x \in Q\}$ exists. In fact, this is set R of Example 1.3.3. □

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(b) Let $A = \{a, b\}$ which exists by the Axiom of Pair. So $\{x \mid x = a \text{ or } x = b\} = \{a, b\}$ exists. □

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