Chapter 2. Relations, Functions, and Orderings
2.1. Ordered Pairs—Proofs of Theorems
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Proof. If \(a = a'\) and \(b = b'\) then \(\{a\} = \{a'\}\) and \(\{a, b\} = \{a', b'\}\) so 
\[(a, b) = \{\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\} = (a', b').\]
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Suppose \((a, b) = (a', b')\); that is, suppose \(\{a\} = \{a'\}\) and \(\{a, b\} = \{a', b'\}\). First, if \(a \neq b\) then \(\{a\} \neq \{a', b'\}\) (since \(b' \in \{a', b'\}\) but \(b \notin \{a\}\)) and so \(\{a\} = \{a'\}\) and \(\{a, b\} = \{a', b'\}\). So \(a = a'\) and then we must have \(b = b'\) (since \(b \neq a = a'\)).
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Suppose \((a, b) = (a', b')\); that is, suppose \(\{a\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\}\). First, if \(a \neq b\) then \(\{a\} \neq \{a', b'\}\) (since \(b' \in \{a', b'\}\) but \(b \notin \{a\}\)) and so \(\{a\} = \{a'\}\) and \(\{a, b\} = \{a', b'\}\). So \(a = a'\) and then we must have \(b = b'\) (since \(b \neq a = a'\)). Second, if \(a = b\) then \(\{a\}, \{a, b\}\} = \{\{a\}, \{a, a\}\} = \{\{a\}\}. So \{\{a\}\} = \{\{a'\}, \{a', b'\}\} and we have \(\{a\} = \{a'\}\) and \(\{a\} = \{a', b'\}\). That is, \(a = a'\) and \(a = a' = b'\) or \(a = a'\) and \(b = b'\), as claimed. \(\square\)
Theorem 2.1.2. \((a, b) = (a', b')\) if and only if \(a = a'\) and \(b = b'\).

Proof. If \(a = a'\) and \(b = b'\) then \(\{a\} = \{a'\}\) and \(\{a, b\} = \{a', b'\}\) so \((a, b) = \{\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\} = (a', b').\)

Suppose \((a, b) = (a', b')\); that is, suppose \(\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\}\}. First, if \(a \neq b\) then \(\{a\} \neq \{a', b'\}\) (since \(b' \in \{a', b'\}\) but \(b \notin \{a\}\)) and so \(\{a\} = \{a'\}\) and \(\{a, b\} = \{a', b'\}\). So \(a = a'\) and then we must have \(b = b'\) (since \(b' \neq a = a'\)). Second, if \(a = b\) then \(\{a\}, \{a, b\}\} = \{\{a\}, \{a', a\}\} = \{\{a\}\}. So \(\{a\} = \{a'\}, \{a', b'\}\} and we have \(\{a\} = \{a'\}\) and \(\{a\} = \{a', b'\}\). That is, \(a = a'\) and \(a = a' = b'\) or \(a = a'\) and \(b = b'\), as claimed.