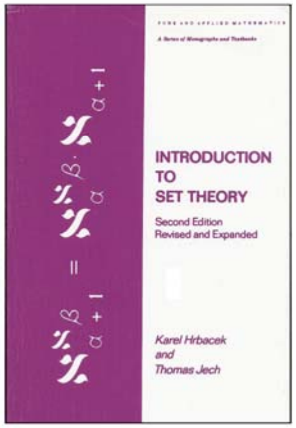
Introduction to Set Theory

Chapter 2. Relations, Functions, and Orderings

2.2. Relations—Proofs of Theorems



Lemma 2.2.9

Lemma 2.2.9. The inverse image of B under R is equal to the image of B under R^{-1}.

Proof. By Exercise 2.2.4(c), $\text{dom}(R) = \text{ran}(R^{-1})$. We have

$$z \in \text{dom}(R) = \{z \mid \text{ there exists } y \text{ such that } xRy\},$$

and so

$$x \in R^{-1}[B] = \{x \in \text{dom}(R) \mid \text{ there exists } y \in B \text{ such that } xRy\}$$

if and only if for some $y \in B$ we have xRy (that is, $(x, y) \in R$). But $(x, y) \in R$ if and only if $(y, x \in R^{-1}$ by definition of R^{-1}. Therefore $x \in R^{-1}[b]$ if and only if for some $y \in B$, $yR^{-1}x$; that is, if and only if $x \in R^{-1}[B]$. So $\text{dom}(R) = \text{ran}(R^{-1})$ equals $R^{-1}[B]$, as claimed. □

Theorem 2.2.A

Theorem 2.2.A. For sets A and B, the cartesian product $A \times B$ exists.

Proof. By Exercise 2.1.1, if $a \in A$ and $b \in B$ then $(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$. So

$$A \times B = \{(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid a \in A \text{ and } b \in B\}.$$

Now $\mathcal{P}(\mathcal{P}(A \cup B))$ exists for given sets A and B by The Axiom of Union and The Axiom of Power Set (applied twice), so $A \times B$ exists by the Axiom Schema of Comprehension. □