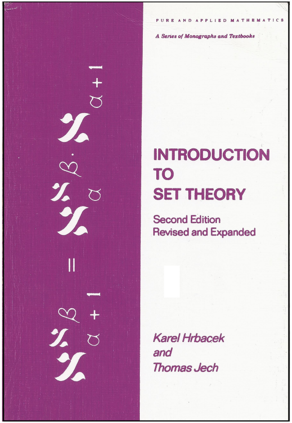
Introduction to Set Theory

Chapter 2. Relations, Functions, and Orderings

2.2. Relations—Proofs of Theorems



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Proof. By Exercise 2.2.4(c), $\text{dom}(R) = \text{ran}(R^{-1})$. We have

$$z \in \text{dom}(R) = \{z \mid \text{ there exists } y \text{ such that } xRy\},$$

and so

$$x \in R^{-1}[B] = \{x \in \text{dom}(R) \mid \text{ there exists } y \in B \text{ such that } xRy\}$$

if and only if for some $y \in B$ we have xRy (that is, $(x, y) \in R$).

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if and only if for some $y \in B$ we have xRy (that is, $(x, y) \in R$). But $(x, y) \in R$ if and only if $(y, x \in R^{-1}$ by definition of R^{-1}. Therefore $x \in R^{-1}[b]$ if and only if for some $y \in B$, $yR^{-1}x$; that is, if and only if $x \in R^{-1}[B]$. So $\mathrm{dom}(R) = \mathrm{ran}(R^{-1})$ equals $R^{-1}[B]$, as claimed. □

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Theorem 2.2.A

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Proof. By Exercise 2.1.1, if $a \in A$ and $b \in B$ then $(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$. So

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