Chapter 2. Relations, Functions, and Orderings

2.2. Relations—Proofs of Theorems
Table of contents

1 Lemma 2.2.9

2 Theorem 2.2.A
Lemma 2.2.9. The inverse image of $B$ under $R$ is equal to the image of $B$ under $R^{-1}$.

Proof. By Exercise 2.2.4(c), $\text{dom}(R) = \text{ran}(R^{-1})$. We have

$$z \in \text{dom}(R) = \{ z \mid \text{there exists } y \text{ such that } xRy \},$$

and so

$$x \in R^{-1}[B] = \{ x \in \text{dom}(R) \mid \text{there exists } y \in B \text{ such that } xRy \}$$

if and only if for some $y \in B$ we have $xRy$ (that is, $(x, y) \in R$).
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if and only if for some $y \in B$ we have $xRy$ (that is, $(x, y) \in R$). But $(x, y) \in R$ if and only if $(y, x) \in R^{-1}$ by definition of $R^{-1}$. Therefore $x \in R^{-1}[b]$ if and only if for some $y \in B$, $yR^{-1}x$; that is, if and only if $x \in R^{-1}[B]$. So $\text{dom}(R) = \text{ran}(R^{-1})$ equals $R^{-1}[B]$, as claimed.
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Theorem 2.2.A. For sets $A$ and $B$, the cartesian product $A \times B$ exists.

Proof. By Exercise 2.1.1, if $a \in A$ and $b \in B$ then $(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$. So

$$A \times B = \{(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid a \in A \text{ and } b \in B\}.$$
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