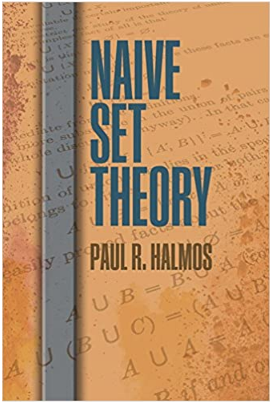
Naive Set Theory

Section 4. Unions and Intersections—Proofs of Theorems



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Next, suppose $x \in (A \cup B) \cap (A \cup C)$. The $x \in A \cup B$ and $x \in A \cup C$. Notice that if $x \notin A$ then we must have both $x \in B$ and $x \in C$; that is, $x \in B \cap C$ and hence $x \in A \cup (B \cap C)$. If $x \in A$, then $x \in A \cup (B \cap C)$. That is, if $x \in (A \cup B) \cap (A \cup C)$ then $x \in A \cup (B \cap C)$, or equivalently $A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$. Therefore, by the Axiom of Extension, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, as claimed. □

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