Section 1. The Axiom of Extension

Note. In this section, we define the equality of two sets in the Axiom of Extension. We consider subset inclusion and briefly mention the properties of reflexive, symmetric/antisymmetric, and transitive.

Note. As with the axiomatic development of geometry, we leave certain terms undefined (they are "primitives") and instead give their relationships to other undefined objects. The properties of these objects are given by the axioms and resulting theorems. For more on this approach, see my online notes for "Introduction to Modern Geometry" (MATH 4157/5157) on Section 1.3. Axiomatic Systems. Our fundamental undefined terms are *set* and *element*.

Note. Informally, we use the term *set* to indicate a "collection' or "family" of objects called *elements* or "members." For example, in geometry a line is a set of points. In addition, a plane is a set of lines (so that a plane it a set of sets). Symbolically, if element x *belongs* (or is *contained* in) set A, we write $x \in A$.

Note. We want to put a relation of *equality* on certain sets. If sets A and B are equal, then we write $A = B$. If they are not equal, we write $A \neq B$. More formally, equality is defined in our first axiom.

Axiom of Extension. Two sets are equal if and only if they have the same elements.

Note. Halmos illustrates the Axiom of Extension by considering an analogous setting where the Axiom of Extension does not hold. Suppose we consider people instead of sets. For x and A people, we write $x \in A$ whenever x is an ancestor of A. The analogous claim to the Axiom of Extension would be that two people are equal if an only if they have the same ancestor. This need not be true, however. When "two" people are equal then they have the same ancestors (the "only if" part), but having the same ancestors does not imply the two people are necessarily equal (i.e., the same; they could be siblings). That is, the "if" part does not hold.

Definition. If A and B are sets and if every element of A is an element of B, the A is a *subset* of B, or B *includes* A, denoted $A \subset B$ or $B \supset A$. If $A \subset B$ and $A \neq B$ then A is a *proper subset* of B.

Note. We do not have a notational way to distinguish between subsets and proper subsets. In other sources, "A is a subset of B" may be denoted $A \subseteq B$ and "A is a proper subset of B" may be denoted $A \subsetneq B$. More confusingly, "A is a proper subset of B" may be denoted $A \subset B$ (the same notation we use for "subset"), so be careful when reading different texts as to which notation they use.

Note. Notice that for every set, $A \subset A$; that is, subset inclusion is *reflexive*. If A, B, and C are sets such that $A \subset B$ and $B \subset C$, then $A \subset C$; that is, subset inclusion is *transitive*. Notice that set equality is also reflexive and transitive.

Note/Theorem 1.A. If A and B are sets such that $A \subset B$ and $B \subset A$, then A and B have the same elements and therefore, by the Axiom of Extension, $A = B$.

Note. The result of Theorem 1.A shows that subset inclusion is *antisymmetric* (as is $\leq$ on the set of real numbers, for example). Set equality is *symmetric*, since $A = B$ implies that $B = A$.

Note. We can reword the Axiom of Extension in terms of subsets as: If A and B are sets, then a necessary and sufficient condition that $A = B$ is that both $A \subset B$ and $B \subset A$. You will repeatedly use this in your mathematical career when showing the equality of two sets.

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