Section 2. The Axiom of Specification

Note. In this section, we discuss making new sets out of old ones. We'll see, in the form of Russell's Paradox, that if we are reckless in our assertions about the existence of sets, we can be lead to contradictions.

Note. As a "heuristic example," let A be the set of all men. The sentence "x is married" is true for some of the elements x of A and false for other. We want to select members of a set that have a certain well-defined property and collect them together in a subset. For example, we could consider the set of all married men, which we denote as $\{x \in A \mid x \text{ is married}\} = \{x \in A : x \text{ is married}\}$ (in these notes we use the former notation, whereas Halmos uses the later notation). Other properties can be considered, such as:

$$\{x \in A \mid x \text{ is married to Kathryn Kreyenbuhl-Gardner}\},$$

in which case *I* am the sole element of the set. Notice that this last set is not *me* but instead is the set with me as an element. Whatever x might be, there is a distinction between x and $\{x\}$. In order to deal with these well-defined properties (and in order to state the Axiom of Specification), we need to introduce the idea of a "sentence."

Definition. An *atomic sentence* is a statement of the form $x \in A$ or $A = B$. A *sentence* (or *condition*) is made of repeated applications of atomic sentences and logical operators.

Note. The logical operators we deal with (expressed, for now, verbally) are: (1) *and*, (2) *or*, (3) *not*, (4) *if—then—* (or *implies*), (5) *if and only if*, (6) *for some* (or *there exists*), and (7) *for all*. We often use parentheses to separate expressions involving multiple logical operators and atomic sentences. We can now state the axiom the gives the existence of subsets.

Axiom of Specification. To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.

Note. By the Axiom of Extension, the set B of the Axiom of Specification is uniquely determined. We denote this set as $B = \{x \in A \mid S(x)\}$.

Note. As an "amusing and instructive application" of the Axiom of Specification (according to Halmos), consider the sentence $S(x)$: not $(x \in x)$. We denote $S(x)$ as $x \notin x$. By the Axiom of Specification $B = \{x \in A \mid x \notin x\}$ is a set. Now

$$y \in B \text{ if and only if } (y \in A \text{ and } y \notin y).$$

Then either $B \in A$ or $B \notin A$. Suppose $B \in A$. Then, either $B \in B$ or $B \notin B$. If $B \in B$, then by the definition of B, $(*)$, we have $B \notin B$, a contradiction. If $B \notin B$ then by $(*)$, the assumption $B \in A$ implies that $B \in B$, again a contradiction. Therefore, $B \in A$ is impossible, and so $B \notin A$. Since A is an arbitrary set, then we see that for every set there is something not in the set. In other words, there is not set containing everything!

Note. Halmos' amusing and instructive application is called "Russell's Paradox." In pre-axiomatic set theory, it was assumed that there was a universal set containing all sets. Russell's Paradox is explored in Mathematical Reasoning (MATH 3000); see my online notes for this class on Section 2.2. Russell's Paradox. Russell's Paradox was motivation for a careful exploration of the foundations of set theory, and resulted in the development axiomatic set theory in the first few years of the 20th century.

Note. In Mathematical Reasoning, Russell's Paradox is explained with the following story, which more illustrates the paradox part of the name. Imagine a town with a barber. The barber cuts the hair of all of those who do not cut their own hair. We ask: "Who cuts the barber's hair?" If the barber does not cut their own hair, then the barber must cut their own hair (since that is their job). If the barbers does cut their own hair, then they cannot cut their own hair since their job is to cut the hair of those who do not cut their own hair. Another, more set theoretic, description is to consider the set B of all sets that are not members of themselves. The question then is: "Is set B a member of itself?" If B is a member of itself, then it cannot me a member of itself since it only consists of such sets. If B is not a member of itself, then it must be a member of itself by its own definition. Therefore such a set cannot exist.

Revised: 1/2/2023