Section 3. Unordered Pairs

Note. In this section, we axiomatically hypothesize that a set exists and that, for given sets a and b, the set $\{a, b\}$ exists. From these two axioms, we can show the existence of infinitely many sets.

Note. To this point, what we have said could vacuously hold if no sets exist. So we add a temporary existence axiom:

> **"Baby Axiom of Existence."** There exists a set.

Note/Definition. Notice that this axiom implies the existence of a set without any elements. If A is a set which exists, then this follows from the Axiom of Specification which implies the existence of the set $\{x \in A \mid x \neq x\}$ since there are no $x \in A$ satisfying $x \neq x$. The Axiom of Extension implies that there is only one set with no elements. We denote this unique set as $\varnothing$, and call the set the *empty set*.

Note. The empty set $\varnothing$ is a subset of every set A: $\varnothing \subset A$ for all sets A. This holds *vacuously*. That is, every element of $\varnothing$ is an of A because there are no elements of A. We now add another axiom. It lets us build a new, nonempty set using existing sets.

Axiom of Pairing. For any two sets, there exists a set that they both belong to. That is, if a and b are sets, then there exists a set A such that $a \in A$ and $b \in A$.

Note. If a and b are sets, we can use the Axiom of Pairing (which gives set A containing a and b) and the Axiom of Specification to show that there is a set that contains only a and b, namely $\{x \in A \mid x = a \text{ or } x = b\}$. By the Axiom of Extension , this set is unique.

Definition. For sets a and b, the set $\{x \in A \mid x = a \text{ or } x = b\}$ above is the *pair* (or *unordered pair*) formed by a and b. We denote this as $\{a, b\}$. For set a, the unordered pair $\{a, a\}$ is a *singleton* of a, denoted $\{a\}$.

Note. With $S(x)$ as the sentence "$x = a$ or $x = b$" (where a and b are sets known to exist), we can express the Axiom of Pairing as the existence of set B such that:

$$x \in B \text{ if and only if } S(x).$$

The Axiom of Specification applied to set A gives the existence of a set B such that:

$$x \in B \text{ if and only if } (x \in A \text{ and } S(x)).$$

All remaining principles of set construction are of the nature of these two applications of the Axiom of Pairing and the Axiom of Specification (in the special case of $S(x)$ used here).

Note. With the known existence of the empty set and by the Axiom of Pairing, we can produce lots of sets: $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$, $\{\{\varnothing\}\}$, $\{\{\{\varnothing\}\}\}$, etc. Notice that each of the sets are different, since they have different elements (this claim is an application of the Axiom of Extension).

Note. As a notational convenience, we mention that we write the set $\{x \mid x \in A \text{ and } S(x)\}$ simply as $\{x \in A \mid S(x)\}$.

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