Section 4. Unions and Intersections

Note. In this section, we introduce two fundamental set operations and give some of their elementary properties.

Axiom of Unions. For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.

Note. With $\mathcal{C}$ as the collection of sets, we can represent the set given by the Axiom of Unions as $U = \{x \mid x \in X \text{ for some } X \text{ in } \mathcal{C}\}$. By the Axiom of Extension, this set is unique.

Definition. The set given by the Axiom of Unions is the *union* of the collection $\mathcal{C}$ of sets. This set is denoted $\cup\mathcal{C} = \cup\{X \mid X \in \mathcal{C}\} = \cup_{X\in\mathcal{C}}X$.

Note. The "simplest facts" about unions, immediately proved from the definitions, include:

$$\cup\varnothing = \cup\{X \mid X \in \varnothing\} = \varnothing,$$

$$\cup\{A\} = \cup\{X \mid X \in \{A\}\} = A.$$

Note 4.A. In the special case of the union of two sets, we have

$$\cup\{X \mid X \in \{A, B\}\} = A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

This allows us to express unions with the logical connective "or." Some easily proved facts about unions of pairs include:

$$A \cup \varnothing = A,$$

$$A \cup B = B \cup A \text{ (commutivity of unions)},$$

$$A \cup (B \cup C) = (A \cup B) \cup C \text{ (associativity of unions)},$$

$$A \cup A = A \text{ (idempotence of unions)},$$

$$A \subset B \text{ if and only if } A \cup B = B.$$

Note. From Note 4.A, we see that

$$\{a\} \cup \{b\} = \{x \mid x \in \{a\} \text{ or } x \in \{b\}\} = \{a, b\}.$$

We also denote the set as $\{a, b\} = \{x \mid x = a \text{ or } x = b\}$. We can then express an *unordered triple* as

$$\{a, b, c\} = \{x \mid x = a \text{ or } x = b \text{ or } x = c\} = (\{a\} \cup \{b\}) \cup \{c\} = \{a\} \cup \{b\} \cup \{c\},$$

where the last term is introduced as an unambiguous representation of the set, since we have associativity of union. We can similarly deal with unordered quadruples, unordered quintuples, or more generally unordered n-tuples.

Definition. For sets A and B, the Axiom of Specification guarantees the existence of the set $A \cap B = \{x \in A \mid x \in B\}$. This set is the *intersection* of A and B.

Note. Notice that $x \in A \cap B$ is and only if x belongs to both A and B, so that

$$A \cap B = \{x \mid z \in A \text{ and } x \in B\} = \{x \in B \mid x \in A\} = B \cap A.$$

Note 4.B. Some easily proved facts about intersections include:

$$A \cap \varnothing = A,$$

$$A \cap B = B \cap A \text{ (commutivity of intersections)},$$

$$A \cap (B \cap C) = (A \cap B) \cap C \text{ (associativity of intersections)},$$

$$A \cap A = A \text{ (idempotence of intersections)},$$

$$A \subset B \text{ if and only if } A \cap B = A.$$

Definition. For sets A and B, if $A \cap B = \varnothing$ then A and B are *disjoint* sets. If every pair of sets in a collection of sets are disjoint, then the collection of sets is *pairwise disjoint.*

Theorem 4.A. For sets A, B, and C we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \text{ and } A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

These are the *distributive laws* of intersection over union and union over intersection, respectively.

Note. Another set equation is given in Exercise 4.1: A necessary and sufficient condition that $(A \cap B) \cup C = A \cap (B \cup C)$ is that $C \subset A$.

Note. Just as we dealt with an arbitrary union over a collection $\mathcal{C}$ os sets above, we can similarly deal with an arbitrary intersection, as follows.

Definition. With $\mathcal{C}$ as a nonempty collection of sets, define set V such that $x \in V$ if and only if $x \in X$ for every X in $\mathcal{C}$. Symbolically,

$$V = \{x \mid x \in X \text{ for every } X \text{ in } \mathcal{C}\}.$$

Set V is the *intersection* of the collection $\mathcal{C}$ of sets. Set V will be denoted as either $\cap\{X \mid X \in \mathcal{C}\}$ or $\cap_{X \in \mathcal{C}} X$.

Note. By writing the intersection of the collection $\mathcal{C}$ as

$$V = \{x \in A \mid x \in X \text{ for every } X \text{ in } \mathcal{C}\},$$

where A is any set in $\mathcal{C}$, we see that the intersection exists by the Axiom of Specification and that it is unique by the Axiom of Extension.

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