Section 1.2. Properties

Note. Hrbecek and Jech comment: “Our exposition in this section is informal. If the reader would like to see how this topic can be studied from a more rigorous point of view, he [or she] can consult some book on mathematical logic” (page 4).

Note. The basic set-theoretic property is the *membership property*, denoted “G.” We read $X \in Y$ as “X is an element of Y.” Here, X and Y are *variables*. All other set-theoretic properties can be stated in terms of membership with the use of “logical means”: identity, logical connectives, and quantifiers.

Note. We use the identity sign “=” to indicate that two variables denote the same set: $X = Y$.

Note. *Logical connection* can be used to construct more complicated properties from more elementary properties. Examples of logical connectives are “not,” “and,” ”or,” “if. . . then. . . ,” and “if and only if.”

Example 1.2.2. Some properties with logical connectives are:

(a) “$X \in Y$ or $Y \in X$” is a property of X and Y.

(c) “If $X = Y$ then $X \in Z$ if and only if $Y \in Z$” of a property of X, Y, and Z.

(d) “‘X is not an element of X” is a property of X, denotes $X \notin X$.

Note. Another type of logical mean is a *quantifier*. Examples of quantifiers are "for all" and "there exists."

Example 1.2.3(b). A property of X involving quantifiers is "For every $Y \in X$, there is Z such that $Z \in X$ and $Z \in Y$." This is not a property of Y nor of Z since they act, if you will, as "dummy variables."

Example 1.2.4(c). Consider the property "For every X, there is $Y \in X$." This is not a property of any variable; the text says it has "no parameters." A property with no parameters as a *statement*. We can associate a truth value with a statement. By the way, the statement here is false. All mathematical theorems are true statements.

Note. We use boldfaced capital letters to denote statements and properties, along with ("if convenient") a list of the parameters in parentheses. We might denote the property in Example 1.2.3(b) as $\mathbf{A}(X)$. In general, $\mathbf{P}(X, Y, \ldots, Z)$ is a property whose truth value depends on parameters $X, Y, \ldots, Z$ (and possibly others).

Note. We can now "give names" to particular properties. So we give our first formal definition.

Definition 1.2.5. We define "X is a subset of Y," denoted $X \subseteq Y$, as the property "$X \subseteq Y$ if and only if every element of X is an element of Y."

Note. Denote the property "There exists no $Y \in X$" as $\mathbf{P}(X)$. In Section 1.3 we show (a) there exists a set X such that $\mathbf{P}(X)$ (in The Axiom of Existence) and (b) if $\mathbf{P}(X)$ and $\mathbf{P}(X')$ then $X = X'$ (in Lemma 1.3.1). That is, there exists a unique set X such that $\mathbf{P}(X)$; that is, there is a unique set with no elements. We will denote this set as $\varnothing$.

Note. Denote the property "For every U, $u \in Z$ if and only if $U \in X$ and $X \in Y$" as $\mathbf{Q}(A, Y, Z)$. Similar to the previous not, in Section 1.3 we show (1) for every X and Y there is Z such that $\mathbf{Q}(X, Y, Z)$ (in Examples 1.3.3 and 1.3.13) and (b) for every X and Y, if $\mathbf{Q}(X, Y, Z)$ and $\mathbf{Q}(X, Y, Z')$ then $Z = Z'$ (in Lemma 1.3.4). This existence and uniqueness allows us to define *the* set Z satisfying $\mathbf{Q}(X, Y, Z)$ for given X and Y. This will be the set $X \cap Y$ called the *intersection* of X and Y.

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