Section 1.3. The Axioms

Note. We now give several axioms, most of which give the existence of certain sets. This is a start to introducing the *Zermelo-Fraenkel axiomatic system* for set theory. In later sections, we complete the “Z-F Axioms” by giving “The Axiom of Infinity” (in Section 3.1, page 50) and “The Axiom Schema of Replacement” (in Section 7.3, page 144). We extend the set of axioms by adding “The Axiom of Choice” (in Section 9.1, page 168) to give the *Zermelo-Fraenkel with Choice axiomatic system* for set theory (or “Z-F-C” for short).

Note. The collection of all sets we deal with is our *universe of discourse*. We introduce axioms which give the existence of certain sets and axioms which allow us to construct new sets from sets which we already know exist.

The Axiom of Existence. There exists a set which has no elements.

The Axiom of Extensionality. If every element of X is an element of Y and every element of Y is an element of X, then $X = Y$.

Note. We now know of the existence of at least one set by the Axiom of Existence. We can use the Axiom of Extensionality to show that, in fact, at this stage we are guaranteed the existence of only one set.

Note. A claim holds *vacuously* if the claim is made about an empty collection of objects; for example, a claim about the elements of a set which has no elements holds vacuously.

Lemma 1.3.1. There exists only one set with no elements.

Note. Since we now know that there is a unique set with no elements, we can give it a name.

Definition 1.3.2. The (unique) set with no elements is the *empty set* (or *null set*), denoted $\varnothing$.

Note. The next axiom allows us to use a property to create subsets of a given set.

The Axiom Schema of Comprehension. Let $\mathbf{P}(x)$ be a property of x. For any set A, there is a set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

Note. Hrbacek and Jech comment that property $\mathbf{P}(x)$ can depend on other parameters $p, \ldots, q$ so that, by the Axiom Schema of Comprehension, for any set A there is a set B consisting exactly of those $x \in A$ for which $\mathbf{P}(x, p, \ldots, q)$ (page 9).

Note. We can use the Axiom Schema of Comprehension to construct a set consisting precisely of elements in both sets A and B. We start as follows.

Example/Theorem 1.3.3. If P and Q are sets then there is a set R such that $x \in R$ if and only if $x \in P$ and $x \in Q$.

Note. The next result implies that the set of Example 1.3.3 is unique for given P and Q. This will allows us to define *the* intersection $P \cap Q$ in Section 1.4.

Lemma 1.3.4. For every set A and every property $\mathbf{P}(x)$, there is only one set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

Note. The uniqueness given in Lemma 1.3.4 allows us to define *the* set of all $x \in A$ such that $\mathbf{P}(x)$.

Definition 1.3.5. $\{x \in A \mid \mathbf{P}(x)\}$ (read "the set of elements in A which satisfy property $\mathbf{P}$") denotes the set of all $x \in A$ with property $\mathbf{P}(x)$.

Example 1.3.6. The set in Example 1.3.3 is denoted $\{x \in P \mid x \in Q\}$ (since $\mathbf{P}(x)$ denotes "$x \in Q$"). In Section 1.4 we denote this as $P \cap Q$,

Note. The only set we axiomatically know to exist at this stage is the empty set. The next axiom allows us to create setes other than the empty set.

The Axiom of Pair. For any sets A and B there is a set C such that $x \in C$ if and only if $x = A$ or $x = B$.

Note. For given A and B, we show in Exercise 1.3.A that set C is unique so that we can refer to *the* unordered pair $\{A, B\}$. We may refer to "sets" and "elements of sets," but technically everything we deal with is a set (even elements of sets).

Example 1.3.7.

(a) For $A = \varnothing$ and $B = \varnothing$, the Axiom of Pair gives us the set $C = \{\varnothing, \varnothing\} = \{\varnothing\}$. Notice that$\{\varnothing\} \neq \varnothing$ since $\varnothing \in \{\varnothing\}$ but $\varnothing \notin \varnothing$. So we now have the existence of a nonempty set.

(b) For $A = \varnothing$ and $B = \{\varnothing\}$, the Axiom of Pair gives us the set $\{\varnothing, \{\varnothing\}\}$. This example foreshadows our approach to the definition of the natural numbers in Chapter 3.

The Axiom of Union. For any set S, there exists a set U such that $x \in U$ if and only if $x \in A$ for some $A \in S$.

Note. The set U of the Axiom of Union is unique for a given set S (as is to be shown in Exercise 1.3.C) and is called the *union* of S.

Example 3.3(c). Let M and N be sets. Then by Axiom of Pair, set $\{M, N\}$ exists. By the Axiom of Union $\cup S = \cup\{M, N\}$ exists. We denote $\cup\{M, N\}$ as $M \cup N$ and can show that it is unique using the Axiom of Extensionality (see Exercise 1.3.C). Notice that $x \in M \cup N$ if and only if $x \in M$ or $x \in N$.

Definition 1.3.10. A is a *subset* of B if every element of A belongs to B. That is, for every $x \in A$ we have $x \in B$. We denote this as $A \subseteq B$ or $B \supseteq A$.

The Axiom of Power Set. For any set, there exists a set P such that $X \in P$ if and only if $X \subseteq S$.

Note. In Exercise 1.3.D it is to be shown that the set P in the Axiom of Power Set is unique for a given set S. Set P is the *power set* of set S, denoted $\mathcal{P}(S)$.

Example 1.3.12(c). $\mathcal{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. In particular, the power set is a "set of sets" (though, of course, everything is a set so all sets are sets of sets...).

Note. Let $\mathbf{P}(x)$ be a property of x (and, possibly, of other parameters). In Exercise 1.3.E it is to be shown that if there is a set A such that for all x, $\mathbf{P}(x)$ implies $x \in A$, then $\{x \in A \mid \mathbf{P}(x)\}$ exists and is unique.

Example/Theorem 1.3.13.

(a) $\{x \mid x \in P \text{ and } x \in Q\}$ exists.

(b) $\{a \mid x = a \text{ or } x = b\}$ exists.

(c) $\{x \mid x \notin x\}$ does not exist. (This is Russell's Paradox.)

Revised: 2/4/2018