Section 1.4. Elementary Operations on Sets

Note. In this section we formally define four set operations and prove some of their properties.

Note 1.4.A. We defined $A \subseteq B$ in Section 1.3 (see Definition 1.3.10). The property $\subseteq$ is called *inclusion*. The following are easy properties of $\subseteq$:

(a) $A \subseteq B$.

(b) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

(c) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Definition. If $A \subseteq B$ and A does not equal B (denoted $A \neq B$), then A is a *proper subset* of B. Hrbaceck and Jech denote this as $A \subset B$ (or $B \supset A$). However, we want to use a different symbol and denote this as $A \subsetneq B$ or $B \supsetneq A$.

Definition 1.4.1. The intersection of A and B, denoted $A \cap B$, is the set of all x which belong to both A and B. The *union* of A and B, denoted $A \cup B$, is the set of all x which belong to either A or B. The *difference* of A and B, denoted $A - B$, is the set of all $x \in A$ which do not belong to B. The *symmetric difference* of A and B, denoted $A \triangle B$, is $A \triangle B = (A - B) \cup (B - A)$.

Note. $A \cap B$ exists and is unique (by Example.Theorem 1.3.3 and Lemma 1.3.4) and $A \cup B$ exists by the Axiom of Union and is unique by Exercise 1.3.C. $A - B$ exists by Exercise 1.3.1 and is unique by Lemma 1.3.4. Existence and uniqueness of $A \triangle B$ follows by combining these.

Note. We now make several set equality claims. From Note 1.4.A we see that we can show set equality $A = B$ by showing both $A \subseteq B$ and $B \subseteq A$.

Theorem 1.4.A. Let A, B, and C be sets. The:

(1) Commutivity: $A \cap B = B \cap A$ and $A \cup B = B \cup A$.

(2) Associativity: $(A \cap B) \cap C = A \cap (B \cap C)$ and $(A \cup B) \cup C = A \cup (B \cup C)$.

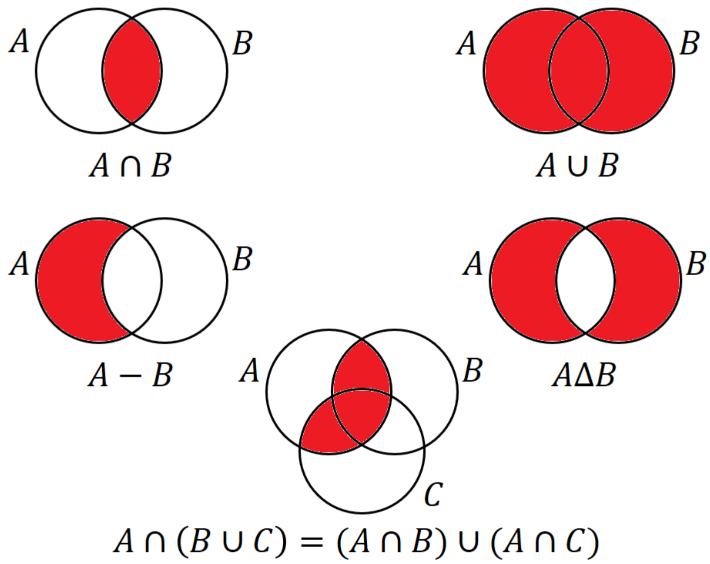
(3) Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(4) De Morgan's Laws: $C - (A \cap B) = (C - a) \cup (C - B)$ and $C - (A \cup B) = (C - A) \cap (C - B)$.

(5) Difference and Symmetric Difference Properties:

- $A \cap (B - C) = (A \cap B) - C$.
- $A - B = \varnothing$ if and only if $A \subseteq B$.
- $A \triangle A = \varnothing$.
- Commutivity: $A \triangle B = B \triangle A$.
- Associativity: $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

Note. We can draw "Venn diagrams" to illustrate certain set properties.



Definition. For nonempty set S, the *intersection* $\cap S$ is the set $\{x \mid x \in A \text{ for all } A \in S\}$.

Note. $\cap S$ exists by Exercise 1.4.6 and is unique by Lemma 1.3.4.

Definition. Sets A and B are *disjoint* if $A \cap B = \varnothing$. A set S is a *system of mutually disjoint sets* if $A \cap B = \varnothing$ for all $A, B \in S$ where $A \neq B$.

Note. The Distributive Law and De Morgan's Law can be generalized to any intersection $\cap S$. Let S and A be sets where $S \neq \varnothing$. Let $T_1 = \{Y \in \mathcal{P}(A) \mid Y = A \cap X \text{ for some } X \in S)$. Then $A \cap (\cup S) = \cup T_1$ (The Generalized Distributive Law). Let $T_2 = \{Y \in \mathcal{P}(A) \mid Y = A - X \text{ for some } X \in S\}$. Then $A - \cup S = \cap T_2$ and $A - \cap S = \cup T_2$ (The generalized De Morgan's Law). These are proved in Exercise 1.4.6.