Section 2.2. Relations

Note. We define unary, binary, and ternary relations, but concentrate on binary relations since these will be used in the next section to define functions.

Definition 2.2.1. A set R is a *binary relation* if all elements of R are ordered pairs; that is, for any $z \in R$ there exists x and y such that $z = (x, y) = \{\{x\}, \{x, y\}\}$. For $(x, y) \in R$, we say x is in relation R with y, denoted xRy.

Definition 2.2.3. Let R be a binary relation.

(a) The set of all x which are in relation R with some y is the *domain* of R, denoted $\text{dom}(R)$. So

$$\text{dom}(R) = \{x \mid \text{ there exists } y \text{ such that } xRy\}.$$

(b) The set of all y such that, for some x, x is in relation R with y in the *range* of R, denoted $\text{ran}(R)$, so

$$\text{ran}(R) = \{y \mid \text{ there exists } x \text{ such that } xRy\}.$$

(c) The set $\text{dom}(R) \cup \text{ran}(R)$ is the *field* of R, denoted $\text{field}(R)$.

(d) If $\text{field}(R) \subseteq X$ then R is a *relation* in X (or R is a *relation between elements of* X).

Note. In Exercise 2.2.1 it is to be shown that $\text{dom}(R)$ and $\text{ran}(R)$ exist.

Example. Let $X = \mathbb{R}$ and $R = \{(x, y) \mid y = x^2\}$ be a relation between the elements of $X = \mathbb{R}$. Then the domain of R is $\text{dom}(R) = \mathbb{R}$ and the range is $\text{ran}(R) = \{x \in \mathbb{R} \mid x \geq 0\}$.

Definition 2.2.3.

(a) The *image* of A under R is the set of all y from the range of R related in R to some element of A, denoted $R[A]$. So

$$R[A] = \{y \in \text{ran}(R) \mid \text{ there exists } x \in A \text{ such that } xRy\}.$$

(b) The *inverse image* of B under R is the set of all x from the domain of R related in R to some element of B, denoted $R^{-1}[B]$. So

$$R^{-1}[B] = \{x \in \text{dom}(R) \mid \text{ there exists } y \in B \text{ such that } xRy\}.$$

Note. In the previous example, $R^{-1}[[1, 4]] = [-2, -1] \cup [1, 2]$.

Definition 2.2.7. Let R be a binary relation. The *inverse* of R is the set

$$R^{-1} = \{z \mid z = (x, y) \text{ for some } x \text{ and } y \text{ such that } (y, x) \in R\}.$$

Note. In the example above, $R^{-1} = \{(x^2, x) \mid x \in \mathbb{R}\}$. Notice that we have not yet addressed "functions" so there is as yet no concern about one-to-one-ness.

Lemma 2.2.9. The inverse image of B under R is equal to the image of B under R^{-1}.

Note. To simplify the notation, we denote

$$\{w \mid w = (x, y) \text{ for some } x \text{ and } y \text{ such that } \mathbf{P}(x, y)\} = \{(x, y) \mid \mathbf{P}(x, y)\}.$$

Definition 2.2.10. Let R and S be binary relations. The *composition* of R and S is the relation

$$S \circ R = \{(x, y) \mid \text{ there exists } y \text{ for which } (x, y) \in R \text{ and } (y, z) \in S\}.$$

Note. We now define some particular relations.

Definition 2.2.11. The *membership relation* on A is denoted

$$\in_Z = \{(z, b) \mid a \in A, b \in B, \text{ and } a \in b\}.$$

The *identity relation* on A is

$$\mathrm{Id}_A = \{(a, b) \mid a \in A, b \in B, \text{ and } a = b\}.$$

Definition 2.2.12. Let A and B be sets. the set of all ordered pairs whose first coordinate is from A and whose second coordinate is from B is the *cartesian product* of A and B, denoted $A \times B$. That is,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Note. The relation $A \times B$ is such that every element of A is related to every element of B. Of course we need to verify that $A \times B$ exists.

Theorem 2.2.A. For sets A and B, the cartesian product $A \times B$ exists.

Definition. For sets A, B, and C, define $A \times B \times C$ as $A \times B \times C = (A \times B) \times C$. Denote an element of $A \times B \times C$ as (a, b, c) where $a \in A$, $b \in B$, and $c \in C$ (though technically $(a, b, c) = ((a, b), c)$).

Definition 2.2.13. A *ternary relation* is a set of ordered triples. That is, S is a ternary relation if for every $u \in S$ there exists x, y, and z such that $u = (x, y, z)$. If $S \subseteq A^3$ then S is a *ternary relation in* A.

Definition. A *unary relation* is a set. A *unary relation in* A is any subset of A.

Note. In Chapter 3, "Natural Numbers," we generalize unary, binary, and ternary relations.