$\mathbf{E = mc^2}$ Turns 100

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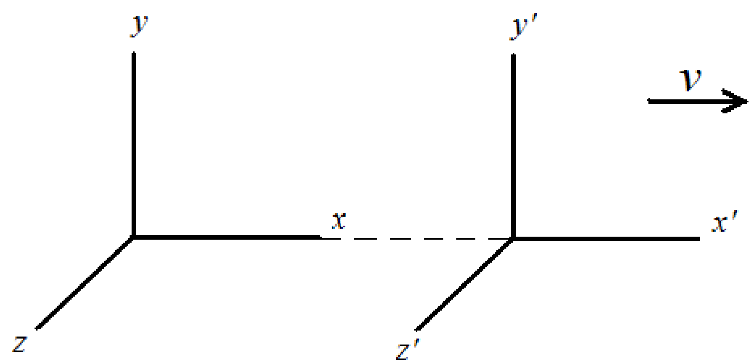
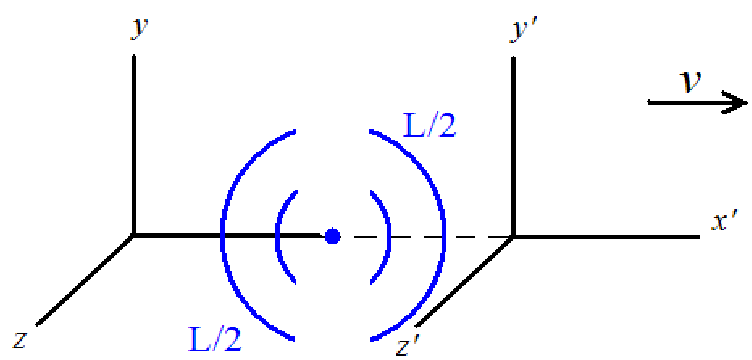
The Setting. Suppose we have inertial frames (x, y, z) and (x', y', z') moving as follows:



Energy in Different Inertial Frames. In Einstein's 1905 papers, he often uses l and L to represent energy. Suppose a system of "plane light waves" moves along the x-axis and has energy l in system (x, y, z) and energy l^* in system (x', y', z'). In "On the Electrodynamics of Moving Bodies" Einstein showed

$$l^* = l\frac{1 - \frac{v}{c}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.$$

Change in Energy. Let there be a body at rest in the system (x, y, z) whose energy in this system is E_0. Let the energy of the body be H_0 relative to system (x', y', z'). Let this body emit plane light waves of energy $L/2$ (measured in the (x, y, z) system) in the positive x direction and at the same time light waves of the same energy in the negative x direction.



This satisfies the Conservation of Energy Principle, so

$$E_0 - E_1 = \frac{L}{2} + \frac{L}{2} = L$$

in the (x, y, z) system, where E_1 is the energy of the body after the emission of the light.

Relativistic Effects. In the (x', y', z') frame,

$$H_0 - H_1 = \frac{L}{2}\frac{1 - \frac{v}{c}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} + \frac{L}{2}\frac{1 + \frac{v}{c}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{L}{\sqrt{1 - \left(\frac{v}{c}\right)^2}},$$

where H_1 is the energy of the body after the emission of the light. Therefore we have

$$(H_0 - E_0) - (H_1 - E_1) = L\left\{\frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - 1\right\}.$$

Kinetic Energy. Einstein then argues physically: "H and E are the energy values of the same body, related to two coordinate systems in relative motion... Hence it is clear that the difference $H - E$ can differ from the body's kinetic energy K with respect to (x', y', z') only by an additive constant C." That is,

$$H_0 - E_0 = K_0 + C$$

$$H_1 - E_1 = K_1 + C$$

since C does not change during the emission of light. Therefore

$$K_0 - K_1 = L\left\{\frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)}} - 1\right\}.$$

Some Details. Now to fill in some details missing from Einstein's brief paper. Let $f(x) = \dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$. Then the MacLaurin series for f is

$$f(x) = 1 + 0x + \frac{x^2}{2} + 0x^3 + \text{(higher order terms)}$$

with radius of convergence 1. Then

$$f\left(\frac{v}{c}\right) = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}.$$

So

$$K_0 - K_1 \approx L\left\{\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) - 1\right\} = \frac{L}{c^2}\frac{v^2}{2}.$$

The Equation. Since kinetic energy is $K = \frac{1}{2}mv^2$, we have

$$K_0 - K_1 = \frac{1}{2}\Delta m v^2 \approx \frac{L}{c^2}\frac{v^2}{2}, \text{ or } \Delta m = \frac{L}{c^2}.$$

That is, $L = mc^2$ where m is "rest mass," or expressing energy as E, $E = mc^2$. ∎