Special Relativity
(Differential Geometry)
Homework, Set 10

1. Use the relationship

$$K = \frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\left[\frac{G_u}{\sqrt{EG}}\right] + \frac{\partial}{\partial v}\left[\frac{E_v}{\sqrt{EG}}\right]\right)$$

to compute the Gauss curvature of a sphere of radius r.

2. Use the relationship of number 1 to compute the Guass curvature of the torus: $\vec{X}(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)$.

3. Let M be an open subset of $\mathbb{R}^2$ with the differentiable structure generated by the identity mapping. Suppose the metric on M is given by $ds^2 = \dfrac{1}{\gamma^2}(du^2 + dv^2)$ where $\gamma = \gamma(u,v)$ is smooth, positive-valued function of the Cartesian coordinates (u,v). Show that M has Guass curvature

$$K = \gamma(\gamma_{uu} + \gamma_{vv}) - (\gamma_u^2 + \gamma_v^2).$$

HINT: When $F = 0$ we have

$$K = -\frac{1}{\sqrt{EG}}\left(\frac{\partial}{\partial u}\left[\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right]\right).$$

Apply this.

4. (*The Poincare Upper Half-Plane*) Suppose that

$$M = \{(u,v) \mid v > 0\},\ \gamma(u,v) = v/k$$

where k is a positive constant. Show that $K = -1/k^2$. HINT: Use number 3.

BONUS. A geodesic of the Poincare Upper Half-Plane satisfies

$$\frac{du}{dv} = \frac{hv}{(k^2 - h^2v^2)^{1/2}}$$

for some constant h. Use this to describe geodesics of the Poincare Upper Half-Plane.