Special Relativity
(Differential Geometry)
Homework, Set 5

1. Let $\vec{\alpha}(t)$ be a smooth curve in E^3, where t is an arbitrary parameter. Let $v(t) = ds/dt$, the speed at parameter value t. Then $\vec{\alpha}\prime(t) = \frac{d\vec{\alpha}}{ds}\frac{ds}{dt} = v\vec{T} = \|\vec{\alpha}'(t)\|\vec{T}$ and $\vec{T}'(t) = \frac{d\vec{T}}{ds}\frac{ds}{dt} = kv\vec{N}$. Show that $k = \|\vec{\alpha}' \times \vec{\alpha}''\|/\|\vec{\alpha}'\|^3$. (Primes signify differentiation with respect to t here. HINT: You may assume that α' and α'' are orthogonal.) BONUS: Do without assuming the orthogonality.

2. Show that the plane curve $\vec{\alpha}(t) = (x(t), y(t))$ has curvature

$$k(t) = \left|\frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}\right|.$$

at $\vec{\alpha}(t)$.

3. As a special case of number 2, show that the graph of $y = f(x)$ has curvature

$$k(x) = \left|\frac{f''(x)}{(1 + f'(x)^2)^{3/2}}\right|$$

at $(x, f(x))$.

4. Let $\vec{\alpha}(t) = (a\cos t, b\sin t)$, $0 \le t \le 2\pi$. Since $x^2/a^2 + y^2/b^2 = 1$, the image of $\vec{\alpha}$ is an ellipse. Compute its curvature $k(t)$ using the formula of number 2 at $t = 0$ and $t = \pi/2$. Sketch the ellipse $x^2/4 + y^2 = 1$ and its osculating circles at the points $(2, 0)$ and $(0, 1)$.