Special Relativity
(Differential Geometry)
Homework, Set 6

1. Consider a torus generated by revolving a circle of radius r, with center $(R, 0)$ about the y-axis (where $R > r$). Find the Gauss curvature of the torus (a) at any point on the "inner equator" of the torus, and (b) at any point on the "outer equator" of the torus. You may argue somewhat informally (by the methods of Section 1.2).

2. For the surface of revolution $\vec{X}(u, v) = (a \cos u \cos v, a \cos u \sin v, b \sin u)$, $-\pi/2 < u < \pi/2$, sketch the profile curve ($v = 0$) in the xz-plane, and then sketch the surface. Prove that $\vec{X}$ is regular and give an equation for the surface in the form $g(x, y, z) = 0$.

3. For the surface of revolution $\vec{X}(u, v) = (a \cosh u \cos v, a \cosh u \sin v, b \sinh u)$, sketch the profile curve ($v = 0$) in the xz-plane, and then sketch the surface. Prove that $\vec{X}$ is regular and give an equation for the surface in the form $g(x, y, z) = 0$.

4. Let $\vec{\alpha}(u) = (\cos u, \sin u, 0)$. Through each point of $\vec{\alpha}(u)$, pass a unit line segment with midpoint $\vec{\alpha}(u)$ and direction vector

$$\vec{\beta}(u) = \left(\sin \frac{u}{2}\right) \vec{\alpha}(u) + \left(\cos \frac{u}{2}\right) (0, 0, 1).$$

The resulting surface

$$\vec{X}(u, v) = \vec{\alpha}(u) + v\vec{\beta}(u) - \frac{1}{2} \leq v \leq \frac{1}{2}$$

is called a *Möbius strip*. (a) Write the coordinate functions of $\vec{X}(u, v)$. (b) Sketch the rulings of $u = 0, \pi/2, \pi$, and $3\pi/2$. (c) Connect the end points of these rulings by drawing the u-parameter curve $\vec{X}(u, 1/2)$, $0 \leq u \leq \pi/4$.