Special Relativity
(Differential Geometry)
Homework, Set 7

1. Derive the formula $ds^2 = dr^2 + r^2 d\theta^2$ for the differential of arc length in polar coordinates by substituting $x = r\cos\theta$, $y = r\sin\theta$ into the formula $ds^2 = dx^2 + dy^2$ for the differential of arc length in Cartesian coordinates.

2. For the torus given by

$$\vec{X}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)$$

where $0 < r < R$, $u \in [0, 2\pi)$, and $v \in [0, 2\pi)$, compute the matrix (g_{ij}), its determinant g, the inverse (g^{ij}), and the unit normal vector $\vec{U}$.

3. Compute the surface area of the torus of number 2.

4. The right circular cylinder $x^2 + y^2 = R^2$ may be parameterized as

$$\vec{X}(u, v) = \left(R\cos\frac{u}{R}, R\sin\frac{u}{R}, v\right).$$

Compute the metric form ds. (If we endow the uv-plane with the Euclidean metric $ds^2 = du^2 + dv^2$, then the result of this exercise shows that any curve in the uv-plane and its image under $\vec{X}$ on the cylinder have the same length. A smooth mapping, such as $\vec{X}$, which preserves lengths of curves is called a *local isometry.* An *isometry* is a local isometry that is one-to-one and onto.)