Special Relativity
(Differential Geometry)
Homework, Set 8

1. Compute the second fundamental form of the helicoid $\vec{X}(u,v) = (u\cos v, u\sin v, bv)$ where b is a constant.

2. Let M be the hyperbolic paraboloid $z = (y^2 - x^2)/2$, and let P be the origin of E^3. Then $T_P(M)$ is the xy-plane. Show that the normal curvature of M at P in the direction of a unit vector $\vec{v} = (\cos\theta, \sin\theta, 0)$ is $K_n(\vec{v}) = -\cos^2\theta + \sin^2\theta = -\cos 2\theta$. What values of θ give the principal directions?

3. Let $\vec{X}(u^1, u^2) = (x(u^1, u^2), y(u^1, u^2), z(u^1, u^2))$ be a surface. Show that

$$L_{ij} = \det \begin{bmatrix} x_{ij} & y_{ij} & z_{ij} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \Big/ \sqrt{g}$$

where, on the right-hand side, subscripts indicate partial differentiation.

4. Compute the Gauss curvature of the hyperbolic paraboloid $z = (y^2 - x^2)/2$ using the fact that this is a surface of the form $z = f(x, y)$.

5. Show that at each point of a surface of revolution, the two principal directions are along the meridian and parallel through that point. HINT: Use the bonus problem.

Bonus. (a) Show that if (at a certain point) $g_{12} = L_{12} = 0$, then the principal curvatures are L_{11}/g_{11}, L_{22}/g_{22}, and the associated principal directions are $\vec{X}_1$, $\vec{X}_2$, respectively.

(b) Conversely, show that if $\vec{X}_1$ and $\vec{X}_2$ are principal, then $g_{12} = L_{12} = 0$.