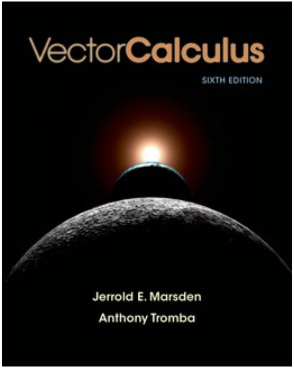
Vector Calculus

Chapter 3. Higher-Order Derivatives; Maxima and Minima

3.5. The Implicit Function Theorem—Proofs of Theorems



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Theorem 3.11. Special Implicit Function Theorem

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Suppose that $F : \mathbb{R}^{n+1} \to \mathbb{R}$ has continuous partial derivatives. Denoting points in $\mathbb{R}^{n+1}$ by $(\mathbf{x}, z)$, where $\mathbf{x} \in \mathbb{R}^n$ and $z \in \mathbb{R}$, assume that $(\mathbf{x}_0, z_0)$ satisfies

$$f(\mathbf{x}_0, z_0) = 0 \text{ and } \frac{\partial F}{\partial z}(\mathbf{x}_0, z_0) \neq 0.$$

Then there is a ball U containing $\mathbf{x}_0$ in $\mathbb{R}^n$ and a neighbor V of z_0 in $\mathbb{R}$ such that there is a unique function $z = g(\mathbf{x})$ defined for $\mathbf{x} \in U$ and $z \in V$ that satisfies $F(\mathbf{x}, g(\mathbf{x})) = 0$. Moreover, if $\mathbf{x} \in U$ and $z \in V$ satisfy $F(\mathbf{x}, z) = 0$, then $z = g(\mathbf{x})$. Finally, $z = g(\mathbf{x})$ is continuously differentiable, with the derivative given by

$$\mathbf{D}[g(\mathbf{x})] = -\frac{1}{\frac{\partial F}{\partial z}(\mathbf{x}, z)} \mathbf{D}_{\mathbf{x}}[F(\mathbf{x}, z)] \bigg|_{z=g(\mathbf{x})},$$

where $\mathbf{D}_{\mathbf{x}}[F]$ denotes the (partial) derivative of F with respect to the variable $\mathbf{x}$; that is, we have $\mathbf{D}_{\mathbf{x}}[F] = [\partial F/\partial x_1, \partial F/\partial x_2, \ldots, \partial F/\partial x_n]$.

Theorem 3.11 (continued 1)

Proof.