Analysis of Fixed Points (i.e. *Critical Points*) in a Two Dimensional Nonlinear system

From

Differential Equations,

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THEOREM 13.7

Hypothesis. *Consider the nonlinear system*

$$\frac{dx}{dt} = ax + by + P_1(x, y),$$
$$\frac{dy}{dt} = cx + dy + Q_1(x, y), \tag{13.43}$$

where a, b, c, d, P_1, and Q_1 satisfy the requirements (1) and (2) above. Consider also the corresponding linear system

$$\frac{dx}{dt} = ax + by,$$
$$\frac{dy}{dt} = cx + dy, \tag{13.19}$$

obtained from (13.43) by neglecting the nonlinear terms $P_1(x, y)$ and $Q_1(x, y)$. Both systems have an isolated critical point at $(0, 0)$. Let λ_1 and λ_2 be the roots of the characteristic equation

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \tag{13.22}$$

of the linear system (13.19).

Conclusions

1. The critical point $(0, 0)$ of the nonlinear system (13.43) is of the same type as that of the linear system (13.19) in the following cases:

(i) If λ_1 and λ_2 are real, unequal, and of the same sign, then not only is $(0, 0)$ a node of (13.19), but also $(0, 0)$ is a node of (13.43).

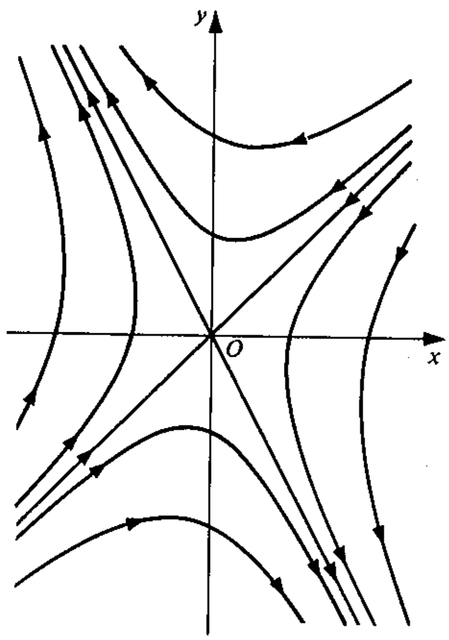
(ii) If λ_1 and λ_2 are real, unequal, and of opposite sign, then not only is $(0, 0)$ a saddle point of (13.19), but also $(0, 0)$ is a saddle point of (13.43).

(iii) If λ_1 and λ_2 are real and equal and the system (13.19) is not *such that $a = d \neq 0$, $b = c = 0$, then not only is $(0, 0)$ a node of (13.19), but also $(0, 0)$ is a node of (13.43).*

(iv) If λ_1 and λ_2 are conjugate complex with real part not zero, then not only is $(0, 0)$ a spiral point of (13.19), but also $(0, 0)$ is a spiral point of (13.43).

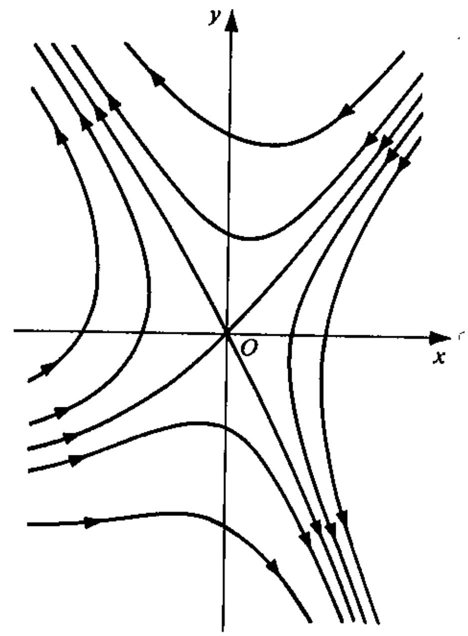
2. The critical point $(0, 0)$ of the nonlinear system (13.43) is not necessarily of the same type as that of the linear system (13.19) in the following cases:

(v) If λ_1 and λ_2 are real and equal and the system (13.19) is such that $a = d \neq 0$, $b = c = 0$, then although $(0, 0)$ is a node of (13.19), the point $(0, 0)$ may be either a node or a spiral point of (13.43).



(a)

Figure 13.13a Linear system.



(b)

Figure 13.13b Nonlinear system.

(*vi*) *If* λ_1 *and* λ_2 *are pure imaginary, then although* (0, 0) *is a center of (13.19), the point* (0, 0) *may be either a center or a spiral point of (13.43).*

Although the critical point (0, 0) of the nonlinear system (13.43) is of the same type as that of the linear system (13.19) in cases (i), (ii), (iii), and (iv) of the conclusion, the actual appearance of the paths is somewhat different. For example, if (0, 0) is a saddle point of the linear system (13.19), then we know that there are four half-line paths entering (0, 0), two for $t \to +\infty$ and two for $t \to -\infty$. However, at the saddle point of the nonlinear system (13.43), in general we have four nonrectilinear curves entering (0, 0), two for $t \to +\infty$ and two for $t \to -\infty$, in place of the half-line paths of the linear case (see Figure 13.13).

Theorem 13.7 deals with the *type* of the critical point (0, 0) of the nonlinear system (13.43). Concerning the *stability* of this point, we state without proof the following theorem of Liapunov.