ANALYSIS OF THE LOGISTIC RECURRENCE RELATION FOR $\lambda \in (1,3)$, $\lambda \neq 2$

Hofbauer and Sigmund Exercise 1.5.1

This solution is from
An Introduction to Chaotic Dynamical Systems,
Second Edition, Robert L. Devaney,
New York: Addison-Wesley Publishing, 1989.

Example 4.10. Let $F_\mu(x) = \mu x(1-x)$ with $\mu > 1$. F_μ has two fixed points: one at 0 and the other at $p_\mu = (\mu - 1)/\mu$. Note that $F'_\mu(0) = \mu$ and $F'_\mu(p_\mu) = 2 - \mu$. Hence 0 is a repelling fixed point for $\mu > 1$ and p_μ is attracting for $1 < \mu < 3$. When $\mu = 3$, $F'_\mu(p_\mu) = -1$. We sketch the graphs of F^2_μ for μ near 3. See Fig. 4.7. Note that 2 new fixed points for F^2_μ appear as μ increases through 3. These are new periodic points of period 2. Another bifurcation has occurred: this time we have a change in $\mathrm{Per}_2(F_\mu)$.

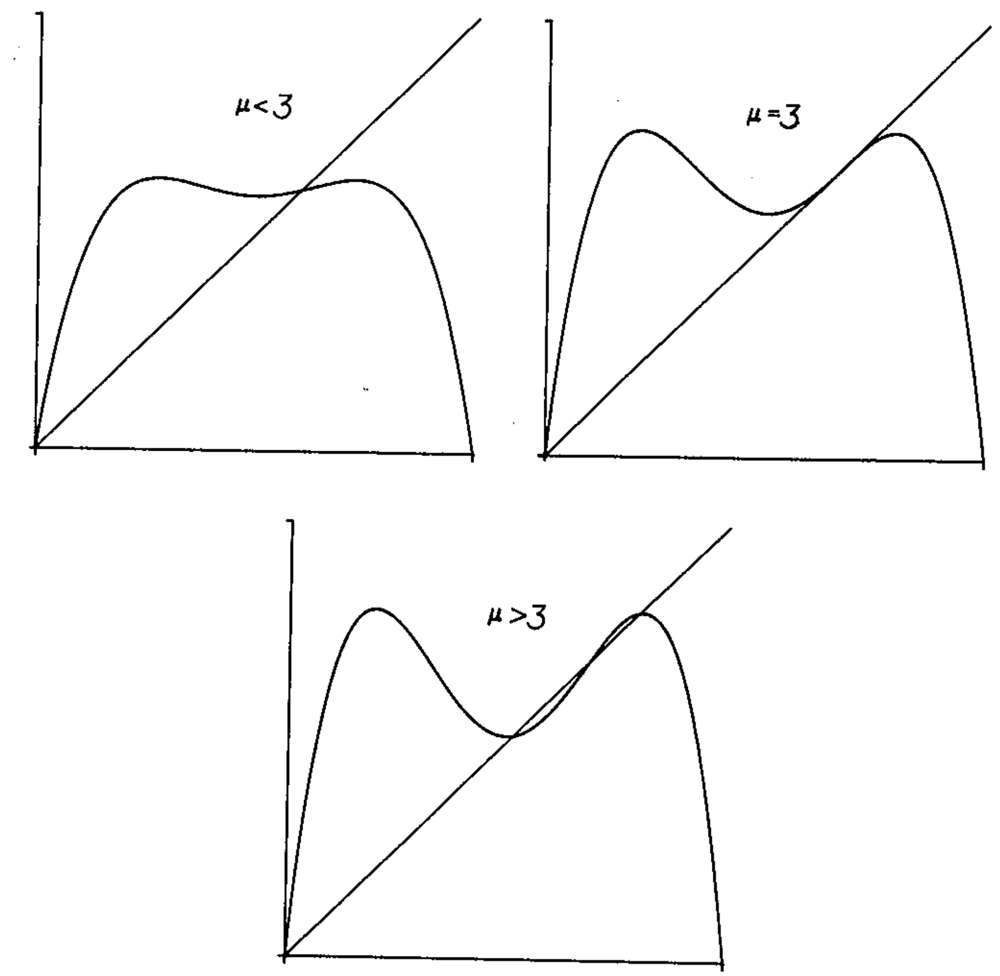

Fig. 4.7. The graphs of $F^2_\mu(x)$ where $F_\mu(x) = \mu x(1-x)$ for $\mu < 3, \mu = 3$, and $\mu > 3$.

Proposition 5.3. *Let $1 < \mu < 3$.*

1. *F_μ has an attracting fixed point at $p_\mu = (\mu - 1)/\mu$ and a repelling fixed point at 0.*
2. *If $0 < x < 1$, then*

$$\lim_{n\to\infty} F_\mu^n(x) = p_\mu.$$

Proof. Part 1 was proved in Example 4.10 at the end of the last section. For part 2, we first deal with the case $1 < \mu < 2$. Suppose x lies in the interval $(0, 1/2]$. Then graphical analysis immediately shows that

$$|F_\mu(x) - p_\mu| < |x - p_\mu|$$

if $x \neq p_\mu$. See Fig. 5.2. Consequently, $F_\mu^n(x) \to p_\mu$ as $n \to \infty$. If, on the other hand, x lies in the interval $(1/2, 1)$, then $F_\mu(x)$ lies in $(0, 1/2)$, so that the previous argument implies

$$F_\mu^n(x) = F_\mu^{n-1}(F_\mu(x)) \to p_\mu$$

as $n \to \infty$.

The case when $2 < \mu < 3$ is more difficult. Graphical analysis shows what is different in this case. See Fig. 5.2. Note that $1/2 < p_\mu < 1$. Let $\hat{p}_\mu$ denote the unique point in the interval $(0, 1/2)$ that is mapped onto p_μ by F_μ. Then the reader may easily check that F_μ^2 maps the interval $[\hat{p}_\mu, p_\mu]$ inside $[1/2, p_\mu]$. It follows that $F_\mu^n(x) \to p_\mu$ as $n \to \infty$ for all $x \in [\hat{p}_\mu, p_\mu]$. Now suppose $x < \hat{p}_\mu$. Again graphical analysis shows that there exists $k > 0$ such that $F_\mu^k(x) \in [\hat{p}_\mu, p_\mu]$. Thus $F_\mu^{k+n}(x) \to p_\mu$ as $n \to \infty$ in this case as well. Finally, as before, F_μ maps the interval $(p_\mu, 1)$ onto $(0, p_\mu)$, so the result follows here as well. Since $(0,1) = (0, \hat{p}_\mu) \cup [\hat{p}_\mu, p_\mu] \cup (p_\mu, 1)$, we are finished. We leave the intermediate case $\mu = 2$ to the reader. See Exercise 1.

q.e.d.

Hence for $1 < \mu < 3$, F_μ has only two fixed points and all other points in I are asymptotic to p_μ. Thus the dynamics of F_μ are completely understood for μ in this range. The phase portraits of F_μ are depicted in Fig. 5.3.

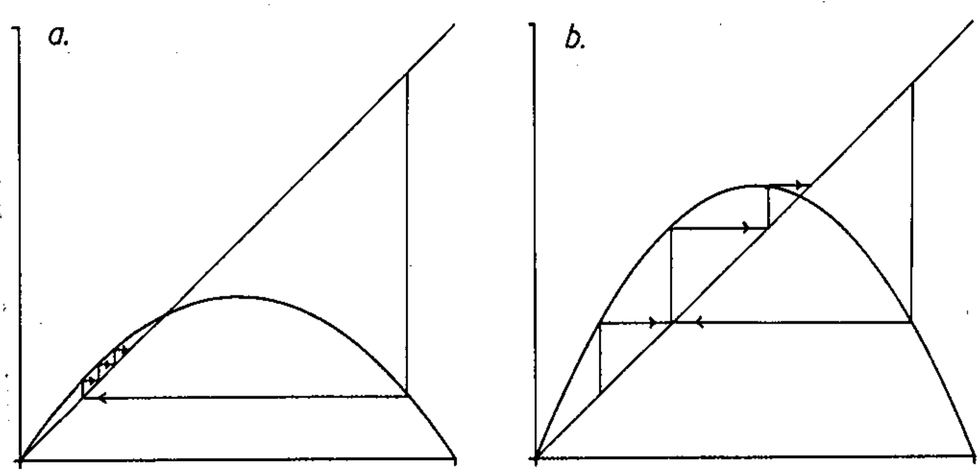

Fig. 5.2. Graphical analysis of $F_\mu(x) = \mu x(1-x)$ when a. $1 < \mu < 2$, and b. $2 < \mu < 3$.

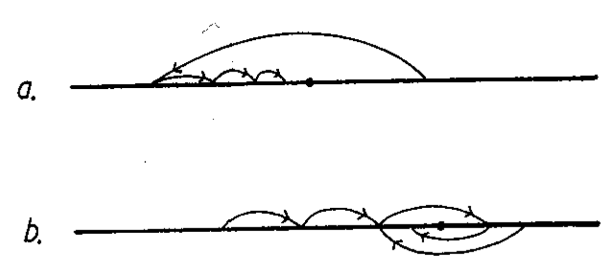

Fig. 5.3. The phase portraits for $F_\mu(x) = \mu x(1-x)$ when a. $1 < \mu < 2$, and b. $2 < \mu < 3$.