A MATHEMATICIAN LOOKS AT CHAOS

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What is Chaos?

"Definition." In the *Proceedings of the IEEE* in an article entitled "Chaos: A Tutorial for Engineers" (**75**(8), 1987) it is stated that "There is no generally accepted definition of chaos. From a practical point of view chaos can be defined as... bounded steady-state behavior that is not an equilibrium point, not periodic, and not quasi-periodic." They then go on to discuss "deterministic systems that exhibit random behavior."

Note. In James Gleick's book *Chaos* (1987), the following are proposed as psuedo-definitions of "chaos:"

1. P. Holmes (mathematician): The complicated, **aperiodic**, attracting orbits of certain dynamical systems.
2. H. Bao-Lin (physicist): A kind of **order** without periodicity.
3. H. B. Stewart (mathematician): **Apparently random** recurrent behavior in a simple deterministic system.
4. R. Jensen (physicist): The irregular, **unpredictable** behavior of deterministic, nonlinear dynamical systems.

Note. We will present a definition of chaos which will take into consideration "random behavior" [or better: **unpredicatability**] (in the form of sensitive dependence on initial conditions), an element of order (in the form of density of periodic points), and an element of "indecomposability."

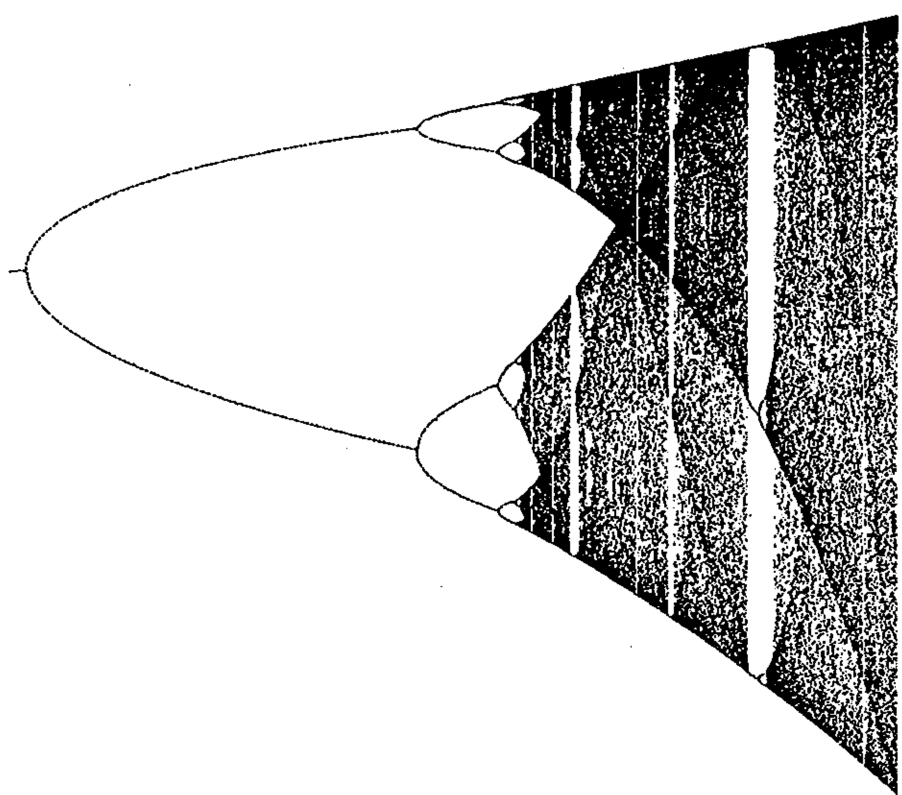


Figure 9.8 Bifurcation diagram of the logistic map (courtesy of J. P Crutchfield)

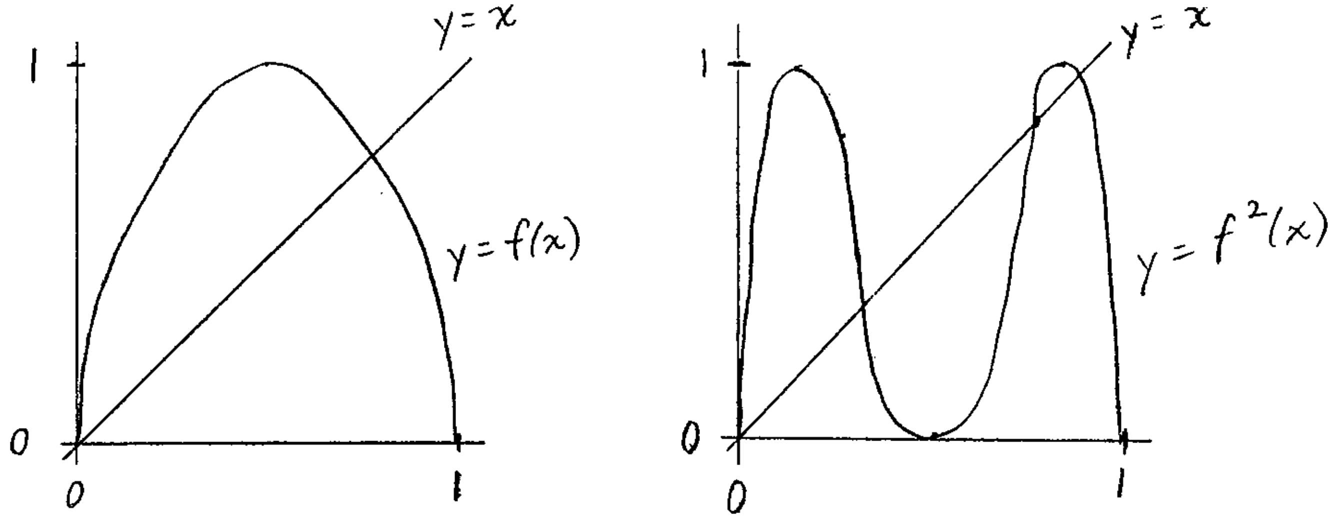
Some Mathematical Background

Definition. A function f defined on a set J such that f maps J into J forms an *iterated function system.* We denote the *iterates* of point $x_0 \in J$ as

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(f(x_0)) = f^2(x_0) \\ x_3 &= f(f(f(x_0))) = f^3(x_0) \\ &\vdots \\ x_{n+1} &= f(x_n) = f^{n+1}(x_0) \\ &\vdots \end{aligned}$$

Definition. An iterated function system with function f and set J has point $x \in J$ as a *periodic point of f* if for some k, $f^k(x) = x$. The smallest such k is called the *period* of x under the action of f. If $f(x) = x$ then x is said to be a *fixed point* of f. Notice that if x is a fixed point of f^n for some n, then x is a periodic point of f.

Example. The function $f(x) = 4x(1-x)$ (which maps $[0,1]$ into itself) has two fixed points (0 and $3/4$) and two period two points:



Definition. A set $Y \subset X$ is *dense* in a set X if for any open set U such that $U \cap X \neq \emptyset$, then $U \cap Y \neq \emptyset$.

Note. If Y is dense in X, then every element of X is a limit point of Y.

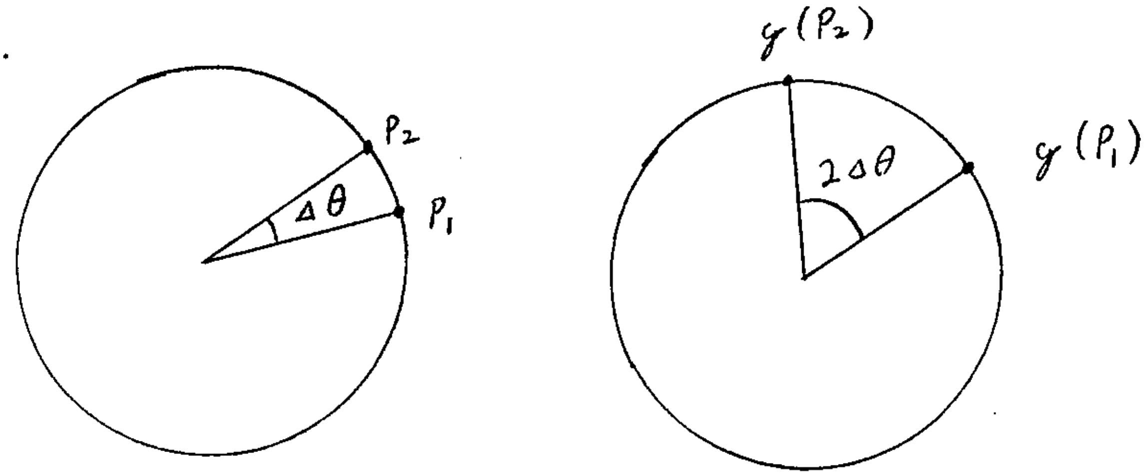
Example. The rational numbers are dense in the real numbers.

Unpredictability: Sensitive Dependence on Initial Conditions

Definition. A function $f : J \to J$ has *sensitive dependence on initial conditions* if:
there exists $\delta > 0$ such that for all $x \in J$ and for any $\epsilon > 0$, there exists $y \in J$ and $n \geq 0$ such that $|x - y| < \epsilon$ and $|f^n(x) - f^n(y)| \geq \delta$.

Note. Intuitively, this means that there is a constant distance δ such that for each x in J, no matter how close to x we look, we can find a $y \in J$ which is seperated from x by a distance of at least δ under the action of f.

Example. Let S^1 denote the unit circle of all points whose polar coordinates are $(r, \theta) = (1, \theta)$. We measure the distance between two points of S^1 as the angular distance between them. Let $g(\theta) = 2\theta$. Then for two points "close together," the angular distance between these points is doubled under the action of g.



Therefore, two points on S^1 which are close together can be made "far apart" and f displays sensitive dependence on initial conditions.

Indecomposability: Topological Transitivity

Definition. A function $f : J \to J$ is *topologically transitive* if for any pair of open sets $U, V \subset J$, there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

Note. Intuitively, this means that open sets are "spread out" under the action of f. Therefore, the system cannot be decomposed into two smaller (open) sets which are invariant under f. This means that you cannot study the behavior of f in some little open subset of J without considering the behavior of f on all of J.

Example. $g(\theta) = 2\theta$ defined on S^1 displays topological transitivity. This can be easily seen since open arcs of S^1 are (rather fundamental) open sets and an arc doubles in length under the action of g. Therefore, for an open arc U and for some k, $g^k(U) = S^1$.

Regularity: Periodic Points

Note. If $f : J \to J$ then we want periodic points to be dense in our definition of chaos.

Example. $g(\theta) = 2\theta$ defined on S^1 has dense periodic points. This can be seen from the fact that the points of the form

$$(r, \theta) = \left(1, \frac{2k\pi}{2^n - 1}\right)$$

for natural number n and for some integer k where $0 \leq k < 2^n$ are periodic. This can be seen from the fact that

$$\begin{aligned}
g^n(\theta) &= g^n\left(\frac{2k\pi}{2^n - 1}\right) \\
&= 2^n\left(\frac{2k\pi}{2^n - 1}\right) \\
&= 2k\pi\left(1 + \frac{1}{2^n - 1}\right) = 2k\pi + \frac{2k\pi}{2^n - 1} \\
&= 2k\pi + \theta \equiv \theta (\text{mod } 2\pi).
\end{aligned}$$

CHAOS!

Definition. Consider the iterated function system with $f : J \to J$. This iterated function system is *chaotic* if

1. f is topologically transitive,

2. f has sensitive dependence on initial conditions, and

3. periodic points of f are dense in J.

(Reference: *An Introduction to Chaotic Dynamical Systems*, R. L. Devaney (1989).)

Note. It has been shown that, in fact, if we have topological transitivity and denseness of periodic points, then we must necessarily have sensitive dependence on initial conditions (Banks *et al.*, *Math. Monthly*, April 1992).

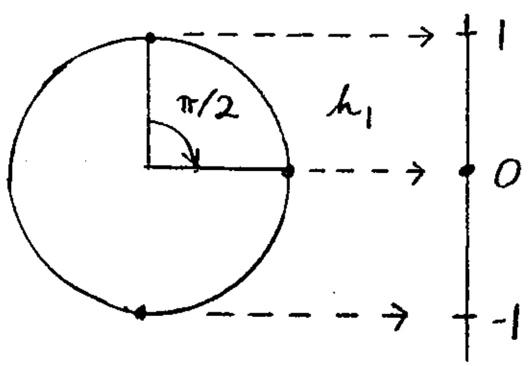
Example. $g(\theta) = 2\theta$ on S^1 is chaotic.

A Detailed Example

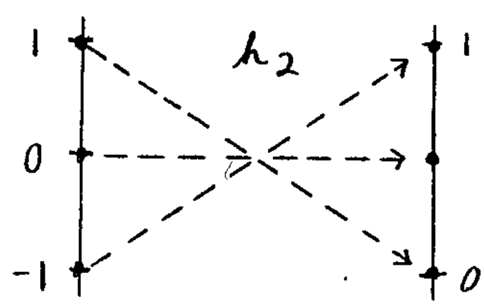
Note. The differential equation $\frac{dx}{dt} = x(K - x)$ is called the *logistic equation.* It describes the growth of a population (of size x) with carrying capacity K. When treated as a difference equation, we have $x_{n+1} = x_n(K - x_n)$. In the terminology of iterated function systems, we have the iterates of the function $f(x) = x(K - x)$.

Theorem. $f(x) = 4x(1 - x)$ is chaotic on $[0, 1]$.

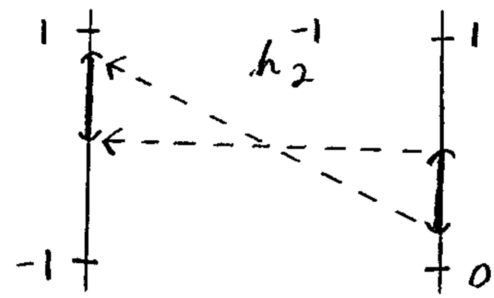
Proof. Define $h_1 : S^1 \to [-1, 1]$ as $h_1(\theta) = \cos \theta$.



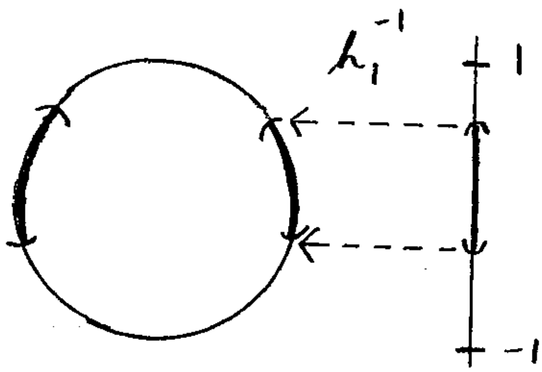
Define $h_2 : [-1, 1] \to [0, 1]$ as $h_2(t) = \frac{1}{2}(1 - t)$.



Since h_1 and h_2 are continuous, inverse images of open sets are open. Let U be an open interval in $[0, 1]$. Then $h_2^{-1}(U)$ is an open interval in $[-1, 1]$.



If neither -1 nor 1 are in $h_2^{-1}(U)$ then $h_1^{-1}(h_2^{-1}(U))$ is two disjoint open arcs in S^1.



Therefore if U is an open interval in $[0, 1]$ (we may assume that neither 0 nor 1 are in U), then there is an open arc u in S^1 such that $h_2 \circ h_1$ maps u one-to-one and onto U (in fact, there are two such u's).

Also, if $h_2 \circ h_1(\theta_0) = \dfrac{1 - \cos\theta_0}{2} = x_0$ (that is, θ_0 "corresponds" to x_0) then

$$
\begin{aligned}
h_2 \circ h_1(\theta_1) &= h_2 \circ h_1(g(\theta_0)) \\
&= h_2 \circ h_1(2\theta_0) \\
&= h_2(\cos(2\theta_0)) \\
&= \frac{1}{2}(1 - \cos(2\theta_0)) \\
&= \sin^2 \theta_0
\end{aligned}
$$

and

$$
\begin{aligned}
x_1 &= f(x_0) = f(h_2 \circ h_1(\theta_0)) \\
&= f(h_2(\cos\theta_0)) \\
&= f\left(\frac{1}{2}(1 - \cos\theta_0)\right) = f\left(\frac{1 - \cos\theta_0}{2}\right) \\
&= 4\left(\frac{1 - \cos\theta_0}{2}\right)\left(1 - \frac{1 - \cos\theta_0}{2}\right) \\
&= \sin^2\theta_0 = h_2 \circ h_1(\theta_1)
\end{aligned}
$$

Therefore $h_2 \circ h_1(\theta_1) = x_1$ (that is, θ_1 "corresponds" to x_1). By mathematical induction, $x_n = f^n(x_0)$ corresponds to $g^n(\theta_0) = \theta_n$ for all integers $n \geq 0$. So we have

(a) if $x = h_2 \circ h_1(\theta)$ and if θ has period k under g, then x has period k under f, and

(b) if $h_2 \circ h_1(\mathrm{u}) = U$ (with the notation above) then $h_2 \circ h_1 \circ g^k(\mathrm{u}) = f^k(U)$ (that is, $g^k(\mathrm{u})$ "corresponds" to $f^k(U)$).

1. f is topologically transitive

Let U and V be open sets in $[0,1]$. Since every open set of real numbers is a countable union of disjoint open intervals, we may assume that U and V are open intervals. Then there are open arcs in S^1, u and v, such that $h_2 \circ h_1$ maps u onto U and v onto V. As seen above, $g(\theta) = 2\theta$ is topologically transitive on S^1, therefore there exists k such that $g^k(u) \cap v \neq \emptyset$. Now $h_2 \circ h_1$ maps $g^k(u)$ onto $f^k(U)$ and v onto V. Therefore $f^k(U) \cap V \neq \emptyset$.

2. f **has sensitive dependence on initial conditions.**

Let $\delta = \frac{1}{2}$ and let U be an open subset of $[0, 1]$ Then there is an open arc u in S^1 "corresponding" to U. Since there is a n such that $g^n(u) = S^1$, for this same n we have $f^n(U) = [0, 1]$. Therefore there is $y \in U$ with $|f^n(x) - f^n(y)| \geq \delta = \frac{1}{2}$.

3. Periodic points of f **are dense in** $[0, 1]$**.**

Let U be an open interval in $[0, 1]$ and let u be as above. Since periodic points of g are dense in S^1, there exists $(1, \theta) \in S^1$ which is periodic under g. Then $h_2 \circ h_1(\theta) = x \in U$ is periodic under f.

Therefore, f is chaotic on $[0, 1]$. ∎