Euclidean Geometry

Euclid (325 BCE – 265 BCE)

Note. (From *An Introduction to the History of Mathematics*, 5th Edition, Howard Eves, 1983.) Alexander the Great founded the city of Alexandria in the Nile River delta in 332 BCE. When Alexander died in 323 BCE, one of his military leaders, Ptolemy, took over the region of Egypt. Ptolemy made Alexandria the capitol of his territory and started the University of Alexandria in about 300 BCE. The university had lecture rooms, laboratories, museums, and a library with over 600,000 papyrus scrolls. Euclid, who may have come from Athens, was made head of the department of mathematics. Little else is known about Euclid.

The eastern Mediterranean from

"The World of the *Decameron*" website.

Euclid's *Elements* consists of 13 books which include 465 propositions. American high-school geometry texts contain much of the material from Books I, III, IV, VI, XI, and XII. No copies of the *Elements* survive from Euclid's time. Modern editions are based on a version prepared by Theon of Alexandria, who lived about 700 years after Euclid. No work, except for the Bible, has been more widely used, edited, or studied, and probably no work has exercised a greater influence on scientific thinking.

Notes. *The Element's* contains a number of definitions. An attempt to define everything is futile, of course, since anything defined must be defined in terms of something else. We are either lead to an infinite progression of definitions or, equally bad, in a circle of definitions. It is better just to take some terms as fundamental and representing something intuitive. This is often the case in set theory, for example, where the terms *set* and *element* remain undefined. We might be wise to take the terms "point" and maybe "line" as such. Euclid, however, sets out to define all objects with which his geometry deals. For historical reasons we reproduce some of these definitions. (All quotes from *The Elements* are from *The Thirteen Books of Euclid's Elements, Translated from the text of Heiberg with Introduction and Commentary* by Sir Thomas L. Heath, Second Edition, Dover Publications, 1956.)

Definitions. From Book I:

1. A *point* is that which has no part.

2. A *line* is breadthless length.

3. The extremities of a line are points.

4. A *straight line* is a line which lies evenly with the points on itself.

5. A *surface* is that which has length and breadth only.

6. The extremities of a surface are lines.

7. A *plane surface* is a surface which lies evenly with the straight lines on itself.

8. A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

9. And when the lines containing the angle are straight, the angle is called *rectilinear*.

10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands.

11. An *obtuse angle* is an angle greater than a right angle.

12. An *acute angle* is an angle less than a right angle.

13. A *boundary* is that which is an extremity of anything.

14. A *figure* is that which is contained by any boundary or boundaries.

15. A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;

16. And the point is called the *center* of the circle.

17. A *diameter* of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

18. A *semicircle* is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

19. *Rectilinear figures* are those which are contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines.

20. Of lateral figures, and *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides alone equal, and a *scalene triangle* that which has its three sides unequal.

21. Further, of trilateral figures, a *right-angled triangle* is that which has a right angle, an *obtuse-angled triangle* that which has an obtuse angle, and an *acute-angled triangle* that which has its three angles acute.

22. Of quadrilateral figures, a *square* is that which is both equilateral and right-angled; an *oblong* that which is right-angled but not equilateral; a *rhombus* that which is equilateral but not right-angled; and a *rhomboid* that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called *trapezia*.

23. *Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Note. Most of these terms are probably familiar to you. However, you may find some of the definitions rather quaint (especially the first seven).

Postulates. Euclid states five postulates. Here are the postulates as stated in *The Elements* and a restatement in more familiar language:

1. *To draw a straight line from any point to any point.* There is one and only one straight line through any two distinct points.

2. *To produce a finite straight line continuously in a straight line.* A line segment can be extended beyond each endpoint.

3. *To describe a circle with any center and distance.* For any point and any positive number, there exists a circle with the point as center and the positive number as radius.

4. *That all right angles are equal to one another.*

5. *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

Note. If you have had a college-level geometry course, then these original postulates may seem to lack rigor. In fact, the terminology is not even very modern. For example, today we distinguish between an angle and its measure. In Euclid, the measure of angles is not discussed, but comparing angles to right angles or two right angles is common.

Note. It is the fifth postulate, and its logical equivalents, with which we have the greatest interest.

Note. *The Elements* contain many assumptions — some stated and some not. Some of the stated assumptions are the five postulates above. Euclid lists other assumptions as "Common Notions" but in contemporary language we would consider them as more postulates (or synonymously, *axioms*).

Common Notions. Euclid's five common notions are a bit clearer than his postulates:

1. Things which are equal to the same thing are also equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be subtracted from equals, the remainders are equal.

4. Things which coincide with one another are equal to one another.

5. The whole is greater than the part.

Note. One of Euclid's assumptions which is not explicitly stated is the fact that he takes lines to be infinite in extent. This is implied by Postulate 3 which guarantees the existence of arbitrarily large circles. Also, in Proposition 12, Euclid refers to an "infinite straight line."

Note. Euclid is interested in establishing the existence of objects. When showing existence, he often gives a constructive way of finding the claimed object. Now let's list the results of Book I and look at a few of Euclid's proofs.

The Propositions of Book I.

Proposition 1. On a given finite straight line to construct an equilateral triangle.

Proposition 2. To place at a given point (as an extremity) a straight line equal to a given straight line.

Proposition 3. Given two unequal straight lines, to cut off from the greater a straight line equal to the less.

Proposition 4. If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

Proposition 5. In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.

Proposition 6. If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

Proposition 7. Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.

Proposition 8. If two triangles have the two sides equal to two sides, respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.

Proposition 9. To bisect a given rectilinear angle.

Proposition 10. To bisect a given finite straight line.

Proposition 11. To draw a straight line at right angles to a given straight line from a given point on it.

Proposition 12. To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.

Proposition 13. If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles.

Proposition 14. If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.

Proposition 15. If two straight lines cut one another, they make the vertical angles equal to one another.

Proposition 16. In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

Proposition 17. In any triangle two angles taken together in any manner are less than two right angles.

Proposition 18. In any triangle the greater side subtends the greater angle.

Proposition 19. In any triangle the greater angle is subtended by the greater side.

Proposition 20. In any triangle two sides taken together in any manner are greater then the remaining one.

Proposition 21. If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.

Proposition 22. Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one.

Proposition 23. On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle.

Proposition 24. If two triangles have the two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base.

Proposition 25. If two triangles have the two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater than the other.

Proposition 26. If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.

Proposition 27. If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

Proposition 28. If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

Proposition 29. A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

Proposition 30. Straight lines parallel to the same straight line are also parallel to one another.

Proposition 31. To draw a straight line through a given point parallel to a given straight line.

Proposition 32. In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

Proposition 33. Straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.

Proposition 34. In parallelogrammic areas the opposite sides and angles equal one another, and the diameter bisects the areas.

Proposition 35. Parallelograms which are on the same base and in the same parallels equal one another.

Proposition 36. Parallelograms which are on equal bases and in the same parallels equal one another.

Proposition 37. Triangles which are on the same base and in the same parallels equal one another.

Proposition 38. Triangles which are on equal bases and in the same parallels equal one another.

Proposition 39. Equal triangles which are on the same base and on the same side are also in the same parallels.

Proposition 40. Equal triangles which are on equal bases and on the same side are also in the same parallels.

Proposition 41. If a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle.

Proposition 42. To construct a parallelogram equal to a given triangle in a given rectilinear angle.

Proposition 43. In any parallelogram the complements of the parallelograms about the diameter equal one another.

Proposition 44. To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle.

Proposition 45. To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

Proposition 46. To describe a square on a given straight line.

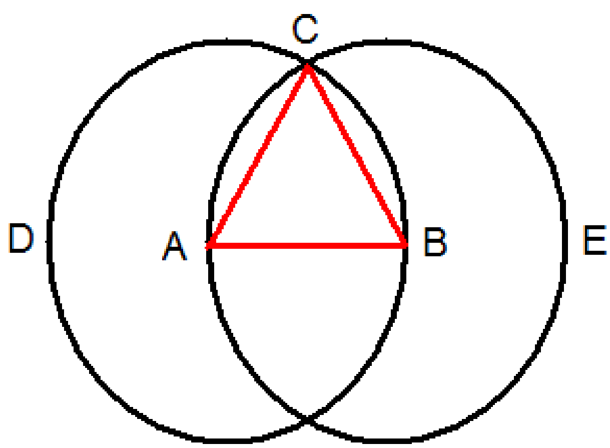
Proposition 47. In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Proposition 48. If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.

Note. The Parallel Postulate is first used in the proof of Proposition 29. Therefore the first 28 propositions do not depend on the Parallel Postulate and may be valid is certain non-Euclidean geometries.

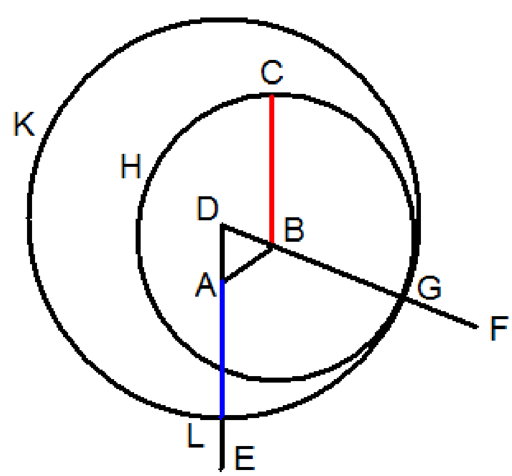
Proposition 1. On a given finite straight line to construct an equilateral triangle.

Proof. Let AB be the given finite straight line. Construct circle BCD with center A and radius AB (Postulate 3). Construct circle ACE with center B and radius AB (Postulate 3). [Here, point C can be defined as a point of intersection of the two circles.] Create line segments AC and BC (Postulate 1). Since A is the center of circle BCD, then AC is equal to AB (Definition 15.) Since B is the center of circle ACE, then AB is equal to BC. So AC is equal to BC (Common Notion 1). Therefore the three straight lines CA, AB, and BC are equal to one another and triangle ABC is an equilateral triangle. QED



Proposition 2. To place at a given point (as an extremity) a straight line equal to a given straight line.

Proof. Let A be the given point and BC the given straight line. Create line segment AB (Postulate 1). Construct equilateral triangle ABD (Proposition 1). Extend line segment DA to line [ray] AE and extend line segment DB to line [ray] BF (Postulate 2). Construct circle CGH with center B and radius BC (Postulate 3). [Here, point G is defined as the intersection of the circle with ray BF.] Construct circle GKL with center D and radius DG (Postulate 3). [Here, point L is the intersection of the circle with ray DE.] Since B is the center of circle CGH, then BC is equal to BG. Since D is the center of circle GKL, then DL is equal to DG. Since DL is composed of DA and AL, and DG is composed of DB and BG, and DA is equal to DB, then the remainder AL is equal to the remainder BG (Common Notion 3). So AL, BC, and BG are equal to each other (Common Notion 1). So AL is equal to BC and AL is the desired line segment. *QED*



Note. Proposition 4 is the familiar side-angle-side (S-A-S) theorem for congruent triangles.

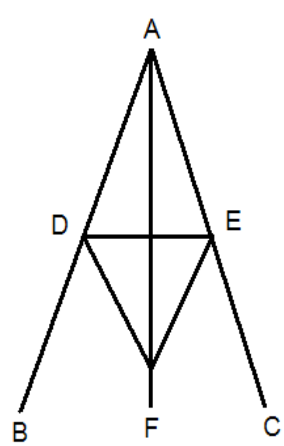
Proposition 4. If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend. (That is, if two triangles have two equal pairs of sides and the two angles between these sides are equal, then the triangles are equivalent or *congruent* — this is often called side-angle-side, S-A-S.)

Proof. Let the two triangles be ABC and DEF, where sides AB and DE are equal, sides AC and DF are equal, and angles BAC and EDF are equal. If triangle ABC is "applied to" (that is, translated to lie on top of) triangle DEF with point A placed on point D, and line segment AB on line segment DE, then point B will coincide with point E because AB is equal to DE. Also, the straight line AC will coincide with line DF because angle BAC is equal to angle EDF. Hence point C will coincide with point F because line segment AC is equal to line segment DF. Therefore base BC coincides with base EF (Common Notion 4). Thus triangle ABC coincides with triangle DEF and corresponding parts are equal. *QED*

Note. The proof of Proposition 4 shows how Euclid interpretes triangles as physical objects which can be moved around (without changing shape — you might say they are "translation invariant"). This is quite different from a "modern" proof.

Proposition 9. To bisect a given rectilinear angle.

Proof. Let the angle BAC be the given rectilineal angle. Take point D on ray AB. Construct point E on segment AC by making line segment AE equal to line segment AD (Proposition 3). Construct line segment DE (Postulate 1). On DE construct equilateral triangle DEF (Proposition 1). Construct ray AF (Postulate 1). Now to show angle BAF equals angle FAC. Consider triangles ADF and AEF. We have AD equal to AE by construction, DF equal to EF by construction, and AF "common" to both triangles. So by Proposition 8 (S-S-S), triangles ADF and AEF are congruent and angles DAF and FAC are equal and hence ray AF bisects angle BAC. *QED*



Note. Proposition 9, again, shows the classical idea of Euclidean construction (sometimes called "straight-edge and compass construction" and inspired by Postulates 1, 2, and 3). Three famous construction problems are:

1. **The Trisection Problem.** Given an angle, cut it into three equal parts.

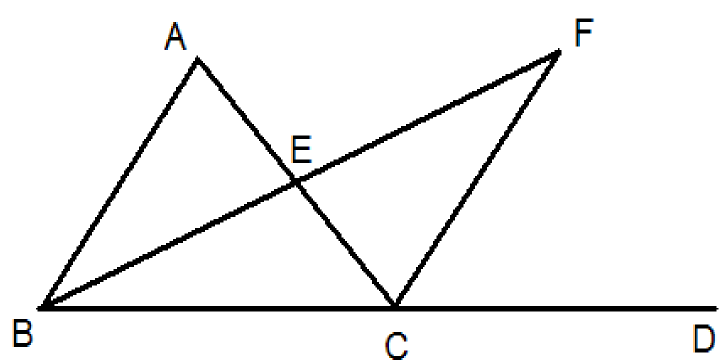
2. **The Duplication of the Cube.** Given a cube of a certain volume, construct the edge of a cube of twice that volume.

3. **The Quadrature of the Circle.** Given a circle of a certain area, construct a square with the same area.

If we restrict ourselves to a finite number of operations with a straight-edge and compass, then no three of the constructions can be accomplished. Surprisingly, the proof that the famous problems *cannot* be solved lies in the theory of algebraic equations and Galois theory.

Proposition 16. In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

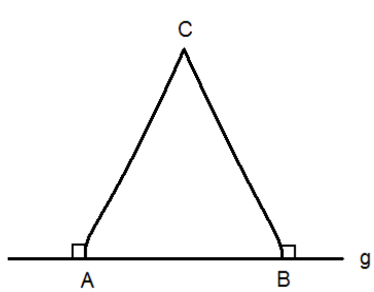
Proof. Let ABC be a triangle and let side BC be extended to point D (Postulate 2) to form exterior angle ACD. Let point E bisect line segment AC (Proposition 10) and construct line segment BC, extending it to point F such that BE equals EF (at *this stage* we are assuming that we can double the length of a line segment and hence are assuming that line segments can be made infinitely long). Create line segment FC (Postulate 1). Consider triangles ABE and CFE. By construction, line segments AE and EC are equal, and line segments BE and EF are equal. Since angles AEB and FEC are vertical angles, they are equal (Proposition 15). So triangles ABE and CFE are congruent (Proposition 4, S-A-S). So angle BAE equals angle ECF. Since angle ACD is greater than angle ACF (Common Notion 5), then angle ACD is greater than angle BAE. Similarly, line segment BC can be bisected to show that angle ACD is greater than angle ABC. *QED*



Note. We can now use Proposition 16 to show that parallel lines exist! When we study non-Euclidean geometry, we will see that there is an example where parallel lines do not exist. Therefore, in this non-Euclidean example, Proposition 16 must not hold. The proof of Proposition 16 will fail at the step where segment BE is doubled in length. In our particular non-Euclidean example (called "Elliptic Geometry"), we will not have lines of infinite length and it will not be possible to double the length of an arbitrary line segment. Much of the following information is from *An Introduction to Non-Euclidean Geometry* by David Gans, Academic Press, 1973.

Theorem. Under Euclid's assumptions (both stated and unstated), excluding Postulate 5, parallel lines exist.

Proof. Let g be any line and A and B two distinct points on g. By Proposition 11 we can construct lines perpendicular to g. We *suppose* these lines meet at some point C forming triangle ABC. But then exterior angle B and opposite interior angle A are both right angles and so are equal. This *contradicts* Proposition 16 and so no such point C exists and the constructed lines are parallel. ∎



Note. In our approach to non-Euclidean geometry, we will appeal to the following result, which is equivalent to the parallel postulate.

Playfair's Theorem. For a given line g and a point P not on g, there exists a unique line through P parallel to g.

Note. If we negate Playfair's "exists a unique," then we have two alternatives: (1) there exists more than one parallel to g through P, and (2) there are no parallels to g through P. This will give us two types of non-Euclidean geometry: (1) hyperbolic, and (2) elliptic, respectively.

Playfair's Existence Theorem. For a given line g and a point P not on g, there exists a line through P parallel to g.

Proof. Let line g and point P be given as described. Let Q be a point on g and create line PQ (Postulate 1). Construct line h through point P so that the angle h makes with PQ is the same as the alternate angle g makes with PQ (Proposition 23). Then by Proposition 27, line h is parallel to line g. ∎

Playfair's Uniqueness Theorem. For a given line g and a point P not on g, there exists a unique line through P parallel to g.

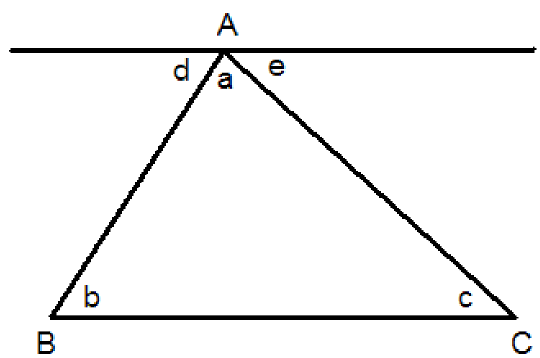
Proof. By Proposition 30, if there were two lines through P parallel to g, then the two lines would be parallel to each other, a contradiction to the fact that both lines pass through P. ∎

Note. Playfair's Existence Theorem does not depend on the Parallel Postulate, and so it will hold in a non-Euclidean system in which Euclid's common notions and first four postulates hold. Since Euclid assumes that lines are infinite in extent then, in a non-Euclidean geometry in which lines are finite, Playfair's Existence Theorem may not hold. This is why Playfair's Existence Theorem does not hold in elliptic geometry.

Note. Let's explore two consequences of the Parallel Postulate.

Theorem (Part of Proposition 32). The sum of the angles of a triangle is equal to two right angles.

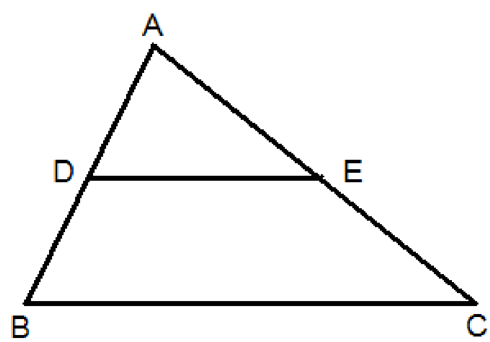
Proof. Consider triangle ABC and create line l parallel to segment $\overline{BC}$. Call the resulting angles l makes with segments $\overline{BC}$ and $\overline{AC}$, d and e, respectively. (Line l exists by Playfair's Existence Theorem — not a consequence of the Parallel Postulate). By Proposition 29 (which follows from the Parallel Postulate), angle d is equal to angle b, and angle e is equal to angle c (alternate angles). Since $a+d+e$ equals two right angles (Proposition 13), then $a+b+c$ equals to right angles. ∎



Note. The previous theorem depends on the Parallel Postulate and does not hold in non-Euclidean geometries. In elliptic geometry, triangles have angles summing to more than two right angles. In hyperbolic geometry, triangles have angles summing to less than two right angles.

Theorem. There exists similar triangles which are not congruent. That is, there exists triangles of the same shape and different sizes.

Proof. Consider a triangle ABC. Let point D be the endpoint of $\overline{AB}$ and point E the midpoint of $\overline{AC}$ (Proposition 10). Construct line segment $\overline{DE}$ (Postulate 1). We can show $\overline{DE}$ is parallel to $\overline{BC}$ (using Cartesian coordinates and slopes, say, which assume the Parallel Postulate). Then by Proposition 29, angle ADE and angle ABC are equal; angle AED and angle ACB are equal. Therefore the angles of triangle ABC are equal to the corresponding angles of triangle ADE. The sides of triangle ABC are twice as big as the corresponding sides of triangle ADE (an argument must be made that $\overline{BC}$ is twice as big as $\overline{DE}$.). ∎



Note. The previous theorem does not hold in elliptic geometry nor in hyperbolic geometry. Surprisingly, we will see that the size of a triangle determines the sum of the angles — "little" triangles have angles summing to near two right angles and "large" triangles have angles which sum to much more (in elliptic geometry) or much less (in hyperbolic geometry) than two right angles.

Note. As we have seen, we could take Playfair's Theorem instead of Euclid's Parallel Postulate and still generate the usual Euclidean geometry. Some other assertions which are equivalent to the Parallel Postulate are:

1. Through a point not on a given line there passes not more than one parallel to the line.
2. Two lines that are parallel to the same line are parallel to each other.
3. A line that meets one of two parallels also meets the other.
4. If two parallels are cut by a transversal, the alternate interior angles are equal.
5. There exists a triangle whose angle-sum is two right angles.
6. Parallel lines are equidistant from one another.
7. There exist two parallel lines whose distance apart never exceeds some finite value.
8. Similar triangles exist which are not congruent.
9. Through any three noncollinear points there passes a circle.
10. Through any point within any angle a line can be drawn which meets both sides of the angle.

11. There exists a quadrilateral whose angle-sum is four right angles.

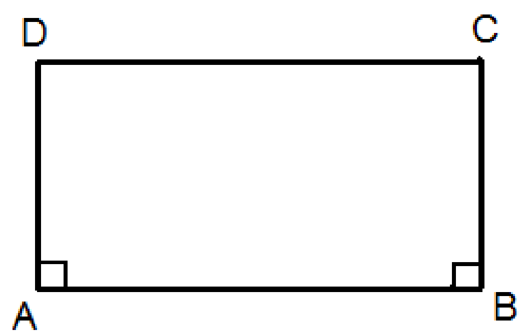
12. Any two parallel lines have a common perpendicular.

By taking Euclid's assumptions (stated and unstated), the first four postulates, and any one of the above, we are lead to the usual Euclidean geometry.

Note. Over the centuries, a number of attempts were made to *prove* the Parallel Postulate based on more elementary properties. It was thought that the Parallel Postulate *had* to be fundamentally true and should not have to be assumed but should be proven like the propositions. Those of us studying math at some point during the last 150 years are quite fortunate to know that Euclid's Parallel Postulate is just one of three possible options concerning parallels.

Note. An influential and significant attempt to prove the Parallel Postulate is due to Gerolano Saccheri. He published (in Latin) *Euclid Freed from Every Flaw* in 1733 (a part is reprinted in *A Source Book on Mathematics* by D.E. Smith, 1959, Dover Publications). Saccheri assumed something contrary to the Parallel Postulate and looked for a contradiction. In the process, he derived several valid results in non-Euclidean geometry.

Note. Saccheri considered a quadrilateral $ABCD$ in which $\overline{AD}$ and $\overline{BC}$ are the same length and are both perpendicular to $\overline{AB}$. He proved correctly that these assumptions imply that angles ADC and BCD are equal. These angles are called the *summit angles*. We will start our study of non-Euclidean geometries by making assumptions about the summit angles.



Note. Proposition 29 of Euclidean geometry implies that the summit angles are right angles.

Note. First, Saccheri assumed the summit angles were larger than right angles (called the *hypothesis of the obtuse angle*). He correctly proved that this implies:

1. $AB > CD$.
2. The sum of the angles of a triangle is greater than two right angles.
3. An angle inscribed in a semicircle is always obtuse (i.e. greater than a right angle).

We might think of the quadrilateral as:

Saccheri did derive contradictions to some of Euclid's propositions. Namely, he found contradictions to Propositions 16, 17, and 18. However, as commented above, these propositions use Euclid's unstated assumptions that lines are infinite in extent. Saccheri had made a tentative exploration of elliptic geometry in which the above three properties hold, and lines are *not* infinite.

Note. Next, Saccheri assumed the summit angles were less than a right angle (called the *hypothesis of the acute angle*). He correctly derived:

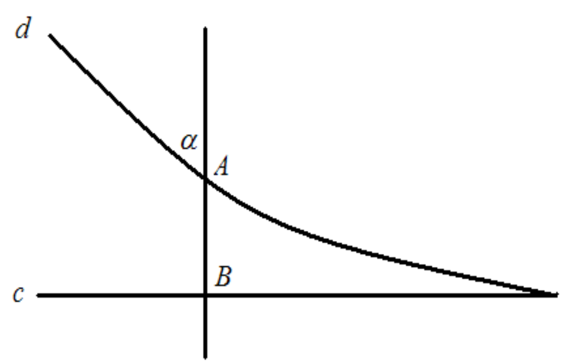
1. $AB < CD$.
2. The angle-sum of every triangle is less than two right angles.
3. An angle inscribed in a semicircle is always acute.
4. If two lines are cut by a transversal so that the sum of the interior angles on the same side of the transversal is less than two right angles, the two lines do not necessarily meet, that is, they are sometimes parallel.

5. Through any point not on a given line there passes more than one parallel to the line.

6. Two parallels need not have a common perpendicular.

7. Two parallels are not equidistant from one another. When they have a common perpendicular, they recede from one another on each side of this perpendicular. When they have no common perpendicular, they recede from each other in one direction and are asymptotic in the other direction.

8. Let c and d be two parallel lines which are asymptotic to the right, A any point on d, and B the projection of A on c. Then α, the angle between d and line AB, is acute, always increases as A moves to the right, and approaches a right angle when A moves without bound in that direction.



From (8), he concluded the lines c and d intersect at an infinitely distant point and took this as a contradiction (remember, this is before Cauchy and so before the idea of limit is formalized). Saccheri's eight conclusions above are valid in the version of non-Euclidean geometry called hyperbolic geometry. However, his "contradiction" was not a contradiction at all. Saccheri thought that he had given a proof of Euclid's Parallel Postulate. Instead, he had opened the door to non-Euclidean geometry and his initial results would inspire others to extend his work into the new area of mathematics called non-Euclidean geometry!