Hyperbolic Geometry

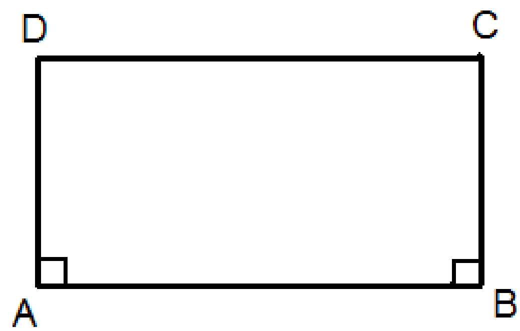
Johann Bolyai
1802–1860

Karl Gauss
1777–1855

Nicolai Lobachevsky
1793–1856

Note. Since the first 28 postulates of Euclid's *Elements* do not use the Parallel Postulate, then these results will also be valid in our first example of non-Euclidean geometry called *hyperbolic geometry*. Recall that one of Euclid's unstated assumptions was that lines are infinite. This will not be the case in our other version of non-Euclidean geometry called *elliptic geometry* and so not all 28 propositions will hold there (for example, in elliptic geometry the sum of the angles of a triangle is always more than two right angles and two of the angles together can be greater than two right angles, contradicting Proposition 17). We will now start adding new propositions (which we call "theorems"), but number them by adding a prefix of "H" for *hyperbolic*.

Note. We will lead into hyperbolic geometry by considering the Saccheri quadrilateral:



We call $\overline{AB}$ the *base*, $\overline{CD}$ the *summit*, $\overline{AD}$ and $\overline{BC}$ the *arms* (which are equal in length), and angles C and D the *summit angles*. We have already stated the following result, but now offer a proof. The following is also valid in Euclidean geometry.

Theorem H29. The summit angles of a Saccheri quadrilateral are equal.

Proof. Triangles ABC and BAD are congruent by Proposition 4 (S-A-S). So $AC = BD$. Then triangles ADC and BCD are congruent by Proposition 8 (S-S-S). Therefore angle ADC equals angle BCD. ∎

Theorem H30. The line joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to each. The base and summit therefore lie on parallel lines having a common perpendicular.

Proof. Let points E and F be the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. The triangles DEA and CBA are congruent by side-angle-side (S-A-S). So $DE = CE$, $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$. So triangles CEF and DEF are congruent by side-side-side (S-S-S). Hence $\angle 5 = \angle 6$, and so each of these angles is a right angle. Also, $\angle 7 = \angle 8$, and so $\angle 1 + \angle 7 = \angle 2 + \angle 8 = 90°$. So line EF is perpendicular to lines AB and CD. Therefore lines AB and CD are parallel by Proposition 27 and EF is the common perpendicular.

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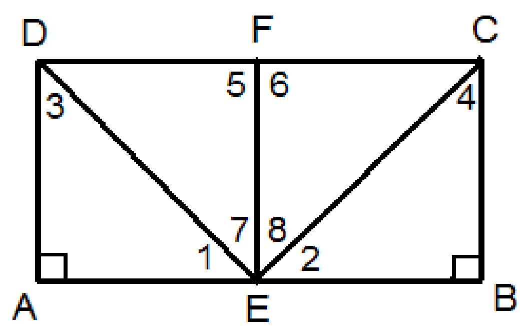
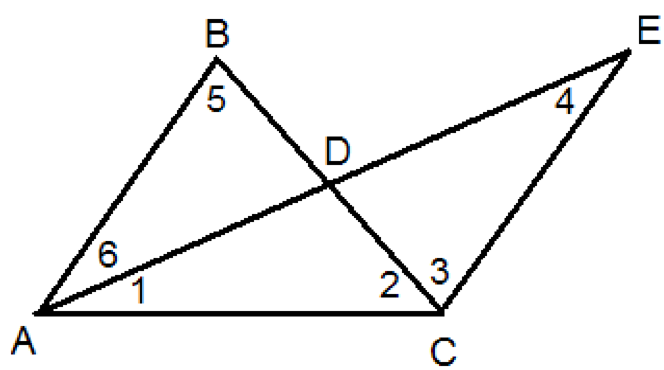


Theorem H31. The angle-sum of a triangle does not exceed two right angles, or $180°$.

Proof. Suppose, to the contrary, that there exists a triangle ABC where the angle-sum is $180° + \alpha$, where α is a positive number of degrees. Let D be the midpoint of $\overline{BC}$ and take E on line AD so that $AD = DE$. (Notice the unstated assumptions that lines are infinite is used here, just as in the proof of Proposition 16.)



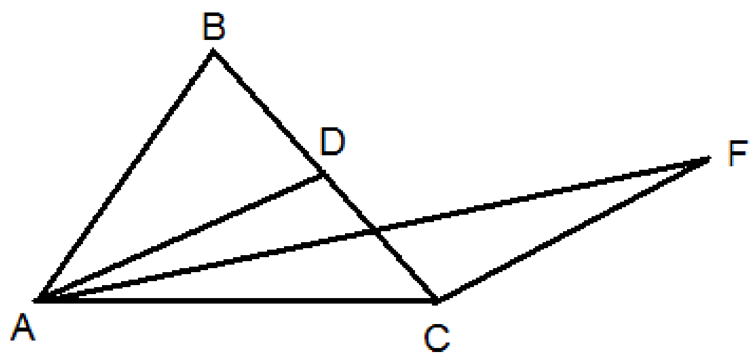
Angles BDA and EDC are vertical angles and so are equal (Proposition 15). So triangles BDA and CDE are congruent by Proposition 4 (S-A-S).

Now the angle-sum of triangle AEC is

$$\angle 1 + \angle 2 + \angle 3 + \angle 4 = \angle 1 + \angle 2 + \angle 5 + \angle 6 \qquad (*)$$

and this is the angle-sum of triangle ABC. Next, notice that $\angle 1 + \angle 6 = \angle BAC$. So either $\angle 1$ or $\angle 6$ is less than or equal to $\frac{1}{2}\angle BAC$. Without loss off generality, suppose $\angle 1 = \angle CAD \leq \frac{1}{2}\angle BAC$. Then we repeat the above process by taking the midpoint of DC and cutting $\angle 1 = \angle CAD$

into $\angle CAF$ and $\angle FAD$.



Now either $\angle CAF \leq \frac{1}{2}\angle CAD$ or $\angle FAD \leq \frac{1}{2}\angle CAD$. That is, either $\angle CAF \leq \frac{1}{2^2}\angle BAC$ or $\angle FAD \leq \frac{1}{2^2}\angle BAC$. Continue in this manner by bisecting the edges of the resulting triangles and creating smaller triangles until a triangle PQR has been constructed with $\angle R < \alpha$. Then in triangle PQR the angle-sum is the same as the angle-sum of triangle ABC by repeated application of $(*)$, and so $180° + \alpha = \angle P + \angle Q + \angle R < \angle P + \angle Q + \alpha$, or $180° < \angle P + \angle Q$. But then triangle PQR is a triangle with two angles summing to more than two right angles, contradicting Proposition 17. ∎

Theorem H32. The summit angles of a Saccheri quadrilateral are not obtuse.

Proof. Suppose, to the contrary, that the summit angles are obtuse. Then the sum of the four angles of the Saccheri quadrilateral exceeds $360°$ and so the angle-sum of either triangle ABC or triangle ACD exceeds $180°$, contradicting Theorem H31. So the summit angles are not obtuse.

∎

Theorem H33. In a quadrilateral with a base, if the arms relative to the base are unequal, so are the summit angles, and conversely, the greater summit angle always lying opposite the greater arm.

Note. Theorems H29–H33 make no assumption about parallel lines and so are valid in both Euclidean geometry and hyperbolic geometry. We now start our study of hyperbolic geometry by explicitly stating our replacement of Euclid's Parallel Postulate.

Postulate H5 (Hyperbolic Parallel Postulate). The summit angles of a Saccheri quadrilateral are acute.

Note. (From *Non-Euclidean Geometry* by Roberto Bonola, Dover Publications, 1955.) Historically, it is recognized that there are three founders of hyperbolic geometry: Carl Frederick Gauss (1777–1855), Nicolai Lobachevsky (1793–1856), and Johann Bolyai (1802–1860). Historical documents (primarily in the form of letters from Gauss to other mathematicians) seem to indicate that Gauss was the first to understand that there was an alterative to Euclidean geometry. Quoting Bonola: "Gauss was the first to have a clear view of geometry independent of the Fifth Postulate, but this remained quite fifty years concealed in the mind of the great geometer, and was only revealed after the works of Lobatschewsky (1829–30) and J. Bolyai (1832) appeared." Based on various passages in Gauss's letters, one can tell that he began his thoughts on non-Euclidean geometry in 1792. However, Gauss did not recognize the existence of a logically sound non-Euclidean geometry by intuition or a flash of genius, but only after years of thought in which he overcame the then universal prejudice against alternatives to the Parallel Postulate. Letters show that Gauss developed the fundamental theorems of the new geometry some time shortly after 1813. However, in his correspondence, Gauss pleaded with his colleagues to keep silent on his results. However, "throughout his life he failed to publish anything on the matter" [*An Introduction to the History of Mathematics*, 5th Edition, by Howard Eves, Saunders College Publishing, 1983]. The honor of being the first to publish work on non-Euclidean geometry goes to others.

Note. (From Eves.) Johann Bolyai was a Hungarian officer in the Austrian army and the son of Wolfgang Bolyai, a math teacher and friend of Gauss. The elder Bolyai published a two volume work on elementary mathematics. In 1829, Johann submitted a manuscript to his father and in 1832 the paper appeared as a 26 page appendix to the first volume of his father's work. The paper was entitled "The Science of Absolute Space" (a translation is available in Bonola). Johann Bolyai never published anything further.

Note. (From Eves and Bonola.) Nicolai Lobachevsky was a Russian math professor at the University of Kasan. On February 12, 1826 (in the old Russian calender) he gave a talk to the Physical Mathematical Section of the University of Kasan. He describes a geometry in which two parallels to a given line can be drawn through a point not on the line. A manuscript of this lecture is not known, but in 1829–30 Lobachevsky published "On the Principles of Geometry" in the *Kasan Bulletin* which contained the ideas of the lecture and further applications. This is the first publication of non-Euclidean geometry and therefore Lobachevsky justifiable has the claim as the first to clearly state properties of non-Euclidean geometry. However, this work received little attention in Russia and practically no attention elsewhere. During the 1830's he published four other works on geometry. Hoping to reach a wider audience, he published "Geometrical Researches on the Theory of Parallels" in 1840 in German (a translation is available in Bonola). In 1855 he published a complete exposition of his geometry in French. Lobachevsky died in 1856 and did not live to see his work receive wide acceptance. For example, Johann Bolyai did not even learn of the existence of Lobachevsky's work until 1848. Some controversy exists as to who deserves credit for the early work in non-Euclidean geometry. "There was considerable suspicion and incrimination of plagiarism at the time" (Eves, page 374). Gauss and Lobachevsky were together in Brunswick, Germany for two years (1805 and 1806) and corresponded later. It is reasonable that Gauss had an influence on Lobachevsky's thought (although Gauss had not solidified his own ideas by 1806). Lobachevsky is given some recognition in that the hyperbolic geometry developed in the first half of the 19th century is sometimes called Lobachevskian geometry.

Note. Since the Hyperbolic Parallel Postulate is the negation of Euclid's Parallel Postulate (by Theorem H32, the summit angles must either be right angles or acute angles). So the negation of anything equivalent to Euclid's Parallel Postulate will be a property of hyperbolic geometry. For example, we can conclude:

(1) Parallel lines exist which are not equidistant from one another.

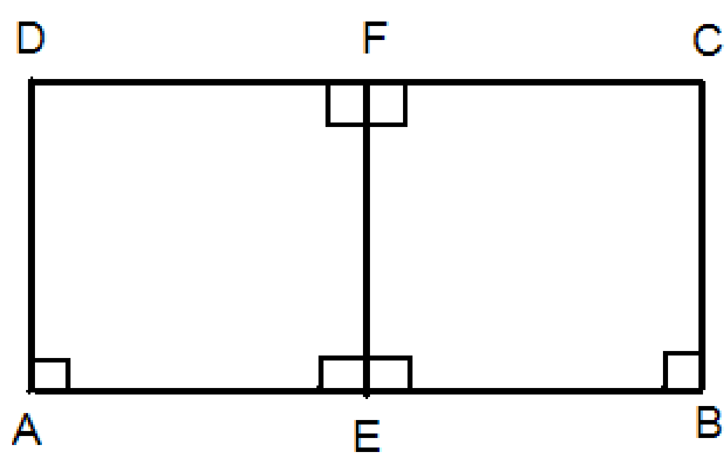
(2) A line and a point not on the line exist such that more than one parallel to the line passes through the point.

(3) Similar triangles are always congruent.

Note. Now we study some properties of hyperbolic geometry which do not hold in Euclidean geometry.

Theorem H34. In a Saccheri quadrilateral, the summit is longer than the base and the segment joining their midpoints is shorter than each arm.

Proof. Let points E and F be the midpoints of the base and summit, respectively. Then $\overline{EF}$ is perpendicular to both the base and summit by Theorem H30. Since $\angle C$ and $\angle D$ are acute, then $AD > EF$ in quadrilateral $AEFD$ and $BC > EF$ in quadrilateral $EBCF$, by Theorem H33. So segment $\overline{EF}$ is shorter than each arm. Now consider quadrilateral $AEFD$ as having arms $\overline{AE}$ and $\overline{DF}$. We have by Theorem H33 that $DF > AE$. Similarly, for quadrilateral $EBCF$, we get $FC > EB$. Hence, combining these two inequalities, $DF + FC > AE + EB$ or $DC > AB$ and the summit is longer than the base. ∎



Note. In the previous construction, we have $\overline{EF}$ as a transversal cutting line $\overline{DC}$ and $\overline{AB}$ with alternate angles equal, and so by Proposition 27, lines $\overline{DC}$ and $\overline{AB}$ are parallel. Since $AD > EF$ and $BC > EF$, we see a specific example of parallel lines which are not equidistant (the definition of distance may seem a little muddled here — we measure the distance from line l to line g *at point* A on line l by measuring the length of line segment $\overline{AD}$ where $\overline{AD}$ is perpendicular to line l and point D is the point where $\overline{AD}$ and g intersect, if in fact they do intersect).

Theorem H35. A *Lambert quadrilateral* is a quadrilateral with three right angles. The fourth angle of a Lambert quadrilateral is acute and each side adjacent to it is longer than the opposite side.

Note. We know by Proposition 27 that *if* two lines have a common perpendicular, *then* they are parallel. Conversely, though, in hyperbolic geometry, parallel lines sometimes have a common perpendicular and sometimes do not.

Theorem H36. If two parallel lines have a common perpendicular, then they cannot have a second common perpendicular.

Proof. If two lines have two common perpendiculars, then they form a Lambert quadrilateral with four right angles, contradicting Theorem H35.

∎

Note. We will need the following two theorems in a proof concerning angle-sums of hyperbolic triangles.

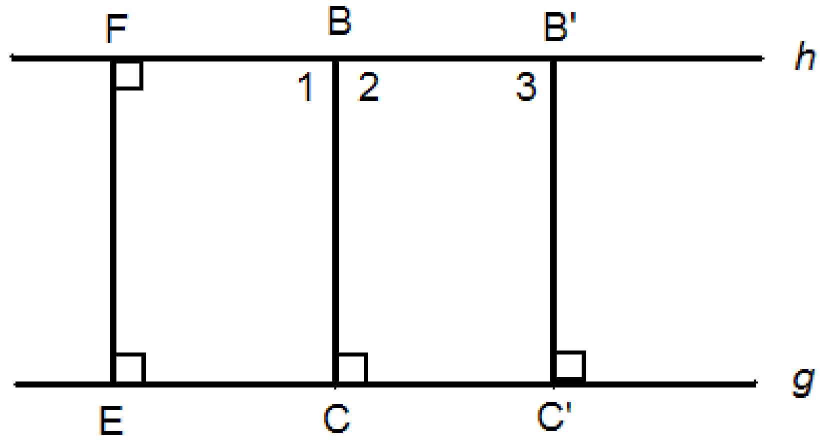
Theorem H37. Two lines will be parallel, with a common perpendicular, if there is a transversal which cuts the lines so as to form equal alternate interior angles or equal corresponding angles.

Theorem H38. If two lines have a common perpendicular, there are transversals which cut the lines so as to form equal alternate interior angles (or equal corresponding angles). The only transversals with this property are those which go through the point on that perpendicular which is midway between the lines.

Theorem H39. The distance between two parallels with a common perpendicular is least when measured along that perpendicular. The distance from a point on either parallel to the other increases as the point recedes from the perpendicular in either direction. (By "distance" from a point to a line is meant the length of a line segment from the line to the point and perpendicular to the line.)

Proof. Let g and h be parallel lines with common perpendicular which meets them at points E and F, respectively. Let B be any point on h other than F and construct a perpendicular to g through B (Proposition

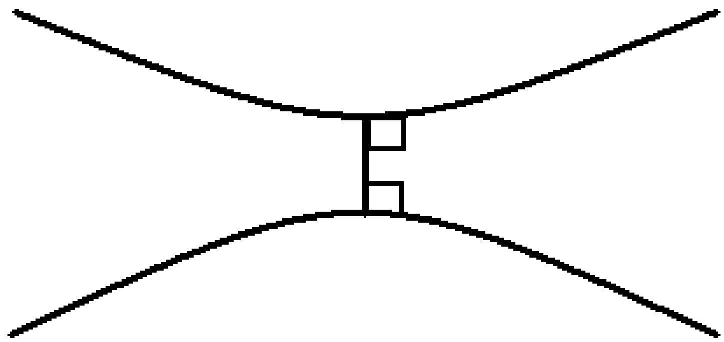
12) and let C be the point where the perpendicular meets g.



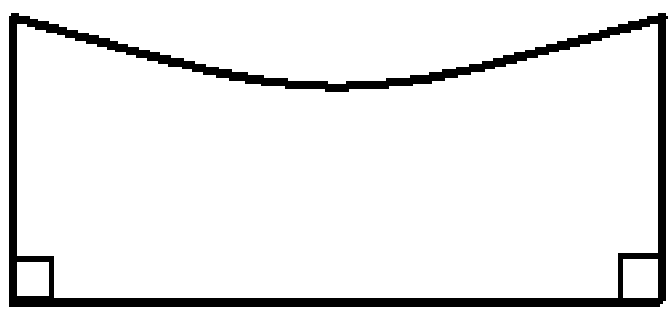
Since $ECBF$ is a Lambert quadrilateral, $\angle 1$ is acute and $BC > FE$ (Theorem H35). So the distance from h to g is less when measured along the common perpendicular than along any other perpendicular from h to g.

Next, choose point B' on h so that B is between F and B'. Let C' be the "projection of B' onto g" (constructed similar to point C above). Then $\angle 3$ is acute since $EC'B'F$ is a Lambert quadrilateral. Also, $\angle 2$ is obtuse since $\angle 1$ is acute and so $\angle 2 > \angle 3$ in quadrilateral $BCC'B'$ and by Theorem H33, $B'C' > BC$ and distance between g and h increases as described. ∎

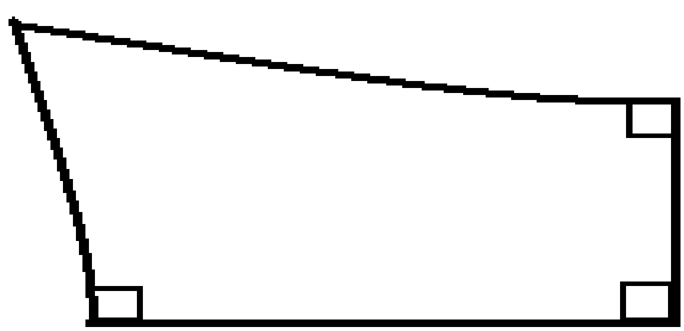
Note. We are now starting to get some insight as to how to visualize parallel lines in hyperbolic geometry. We might think of parallel lines with a common perpendicular as "bending away" from each other:



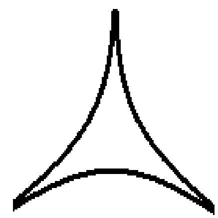
Then a Saccheri quadrilateral is like:



and a Lambert quadrilateral is like:



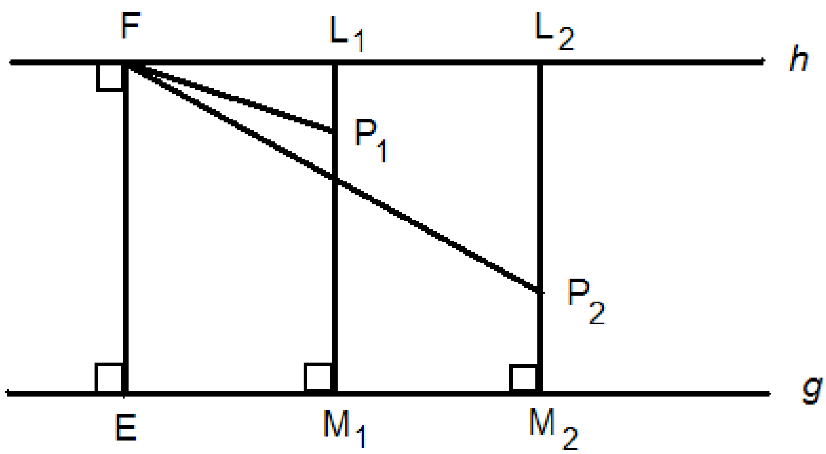
We can think of triangles, then, as looking like:



Note. The following is very clearly in contradiction to the result from Euclidean geometry called Playfair's Theorem.

Theorem H40. Given any line and any point not on it, there passes infinitely many lines through the point which are parallel to the line and have a common perpendicular with it.

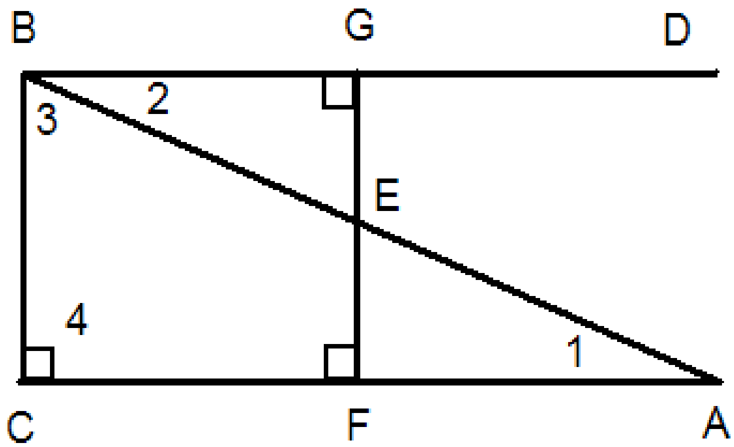
Proof. Let g be a line and F a point not on it. If E is the projection of F on g and line h is perpendicular to line $\overline{EF}$ at F, then h is one line through F which is parallel to g and $\overline{EF}$ is the common perpendicular. Next, take a point L_1 on h to the right, say, of F. Let point M_1 be the projection of L_1 onto line g. Then $L_1M_1 > EF$ by Theorem H39. Let P_1 be the point on $\overline{L_1M_1}$ such that $M_1P_1 = EF$. Then EM_1P_1F is a Saccheri quadrilateral and so line FP_1 and g are parallel and have a common perpendicular by Theorem H30 (the common perpendicular is based on the midpoints of $\overline{FP_1}$ and $\overline{EM_1}$). Similarly, by taking L_2 as a point to the right of F, and create line FP_2 parallel to g. Now line h is distinct from lines FP_1 and FP_2 "clearly." We need to show lines FP_1 and FP_2 are distinct. Suppose to the contrary that lines FP_1 and FP_2 are the same. Then there are three points on this line, F, P_1, and P_2, which are each the same distance from line g, contradicting one of the homework problems. Therefore the lines are distinct and for any point on h to the right (or for that matter, the left) of F, there is a line parallel to g through F. ∎



Note. We now turn our attention to hyperbolic triangles. We have already argued that the angles of a triangle sum to less than 180° by considering negatives of things logically equivalent to Euclid's Parallel Postulate. We now give a proof of this based on other established results.

Theorem H42. The angle-sum of every triangle is less than 180°.

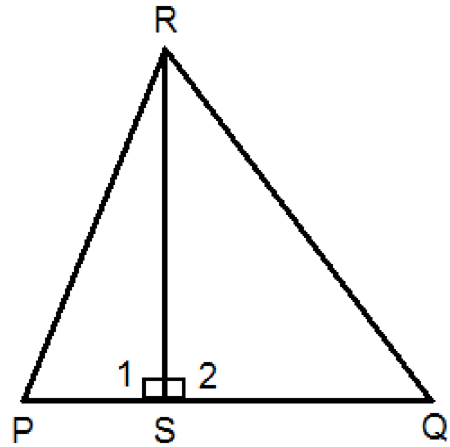
Proof. First, consider a right triangle ABC with right angle C. Let BD be the line through B such that $\angle 1 = \angle 2$.



Lines AC and BD are then parallel by Proposition 27 and have a common perpendicular by Theorem H37. Let the common perpendicular meet lines AC and BD at points F and G, respectively, and let E be the midpoint of $\overline{FG}$. Then by Theorem H38, E is on line AB and $AE = EB$ by one of the homework problems. So $BCFG$ is a Lambert quadrilateral and $\angle CBG$ is acute by Theorem H35. So $\angle 1 + \angle 3 = \angle 2 + \angle 3 = \angle CBG < 90°$, or $\angle 1 + \angle 3 < 90°$. Hence in triangle ABC the angle-sum is $\angle 1 + \angle 3 + \angle 4 < 180°$, and the claim holds for right triangles.

Second, consider any triangle PQR which is *not* a right triangle. By Theorem H31, triangle PQR can have at most one obtuse angle and so must contain at least two acute angles, say at P and Q. Let S be the projection of R onto line PQ. By Proposition 16, S lies between P and

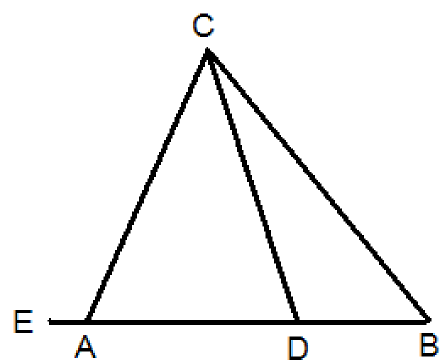
Q. So line RS subdivides $\angle PRQ$ and we have two right triangles PRS and QRS. From above, the sum of the angles in these two triangles is less than $360°$. It follows that the angle-sum of triangle PQR is less than $180°$. ∎



Note. In fact, for any $\sigma \in (0°, 180°)$, there is a triangle with angle-sum σ. Informally, "small" triangles have angle-sums near $180°$ and "large" triangles have angle-sums near $0°$. The following result partly illustrates this.

Theorem H43. There are triangles with angle-sum arbitrarily close to $180°$.

Proof. Consider any triangle ABC and a variable point D between A and B. Let D approach A (here we are using unstated assumptions involving the continuum and limits). Then $\angle ADC$ approaches $\angle EAC$, and $\angle ACD$ approaches 0. Hence as D approaches A, the angle-sum of ACD, $\angle ACD + \angle CDA + \angle CAD$, approaches $0° + \angle CAE + \angle CAD = 180°$. So the angle-sum of ACD can be made arbitrarily close to $180°$ by making D sufficiently close to A. (This could be neated up with ϵ's and δ's.) ∎



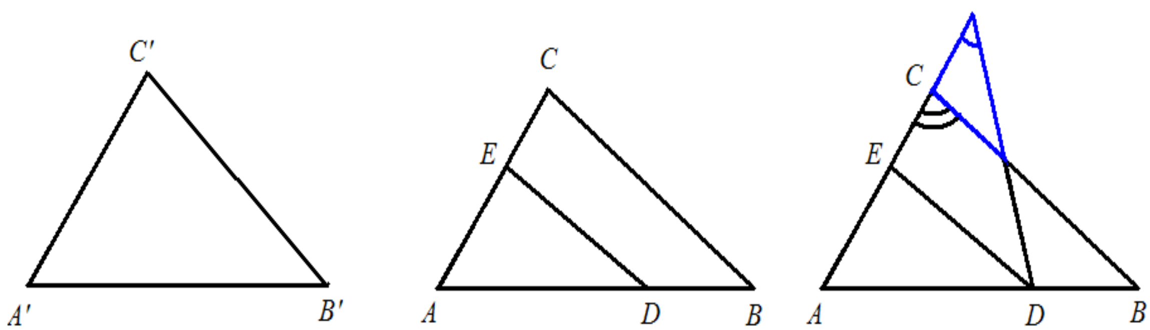
Theorem H44. The angle-sum of every (convex) quadrilateral is less than 360°.

Idea of Proof. We can cut the quadrilateral into two triangles and apply Theorem H43. ∎

Note. The following result is certainly not true in Euclidean geometry. We might call it angle-angle-angle (A-A-A).

Theorem H45. Two triangles are congruent if the three angles of one are equal respectively to the three angles of the other.

Proof. Consider triangle ABC and $A'B'C'$ where the corresponding angles are equal. Suppose $AB > A'B'$. Take point D on $\overline{AB}$ such that $AD = A'B'$. On line AC, on the same side of A as C, take point E such that $AE = A'C'$. Then by construction, triangles $A'B'C'$ and ADE are congruent (Proposition 4, S-A-S). So $\angle ADE = \angle B$ and $\angle AED = \angle C$.



Now if point E is not between points A and C, we have a triangle in which an exterior angle equals an opposite interior angle, contradicting Proposition 16. So point E is between points A and C. Then in quadrilateral $BCED$ the angle-sum equals 360°, contradicting Theorem H44. We conclude that $AB = A'B'$ and similarly $AC = A'C'$ and $BD = B'C'$. Therefore triangles ABC and $A'B'C'$ are congruent. ∎

Note. Let's briefly discuss the area of triangles in hyperbolic geometry. First, define the *defect* of a triangle to be the amount by which its angle-sum is less than 180°. We want to discuss area, but remember that area is thought of in terms of squares and squares do not exist in hyperbolic geometry (this is a homework problem). So instead of thinking of squares giving areas, we deal more with triangles (remember, triangles are congruent when their corresponding angles are equal). With this approach to "area," we find that the area A of a triangle is proportional to its defect D: $A = kD$. This parameter k gives us a fundamental constant for a hyperbolic geometry. Surprisingly, it is easy to tell when two hyperbolic triangles have the same area:

Theorem H54. Two triangles have the same area if and only if they have the same angle-sum.

Note. For the last 100 or 150 years, mathematics has been thought of as an abstract axiomatic system independent of preconceived notions and, especially, independent of the (necessary) use of pictures. Euclidean geometry is a system within itself independent of dots, lines, and circles on paper, as well as the Cartesian plane and Cartesian coordinates. However, we use the usual Cartesian plane as a *model* of Euclidean geometry. We then use the model to help us visualize properties of the axiomatic system. We want a similar model for hyperbolic geometry.

Note. (From *Foundations of Euclidean and Non-Euclidean Geometry*, Richard Faber.) The first model for hyperbolic geometry is due to Eugenio Beltrami (1835–1900). In 1868 he published his *Essay on the Interpretation of Non-Euclidean Geometry* in which he described a surface in Euclidean space which gives a partial representation of the hyperbolic plane. The surface is called a *pseudosphere*. It can be generated as follows. Let a box be located in the xz-plane at position $(1,0)$ with a chain of unit length attached to it. We start with the other end of the chain at the point $(0,0)$ and then move that end up the z-axis, dragging the box along. The path can be described parametrically as

$$(x(t), z(t)) = (\sin t, \cos t + \ln\left(\tan\left(\frac{t}{2}\right)\right),\ t \in \left(0, \frac{\pi}{2}\right].$$

This curve is called a *tractrix* or "drag curve." If we revolve the curve around the z-axis to generate a surface in 3-dimensional Euclidean space, we get the surface (in terms of parameters u and v):

$$(x(u,v), y(u,v), z(u,v)) = \left(\sin u \cos v, \sin u \sin v, \cos u + \ln\left(\tan\left(\frac{u}{2}\right)\right)\right).$$

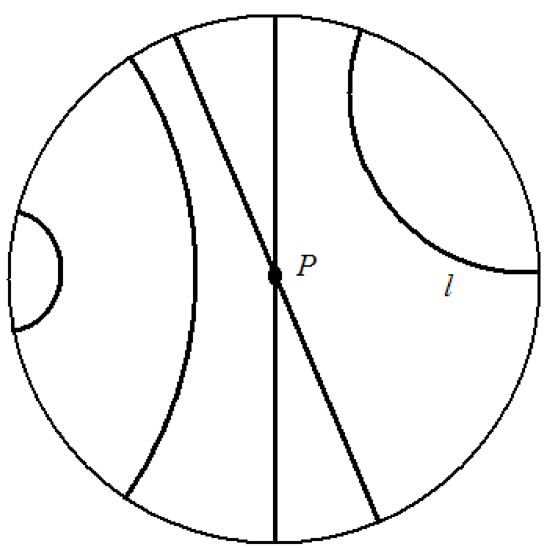
This surface is the pseudosphere.

Note. In the pseudosphere model, "points" are points on the surface, "lines" are geodesics on the surface, and the "plane" is the surface itself. One example of lines on the surface are cross sections which are tractrices. The pseudosphere has one desirable property and one undesirable property. The desirable property is a property of homogeneity — it has uniform *curvature* (the curvature at any interior point is -1). the undesirable property is that it has a boundary and lines cannot be extended indefinitely from all points. By analogy, we could think of taking a piece of the Euclidean half plane and roll it up into a cylinder (Euclidean planes and such cylinders have zero curvature). The pseudosphere is a piece of the hyperbolic plane that has been rolled up into a hyperbolic cylinder (the curvature of the hyperbolic plane and the pseudosphere are both $-1\ldots$ or at least both are a negative constant if we modify things a little). So the pseudosphere is only a "local" model for the hyperbolic plane (not a "global" model). In fact, David Hilbert, in 1901, proved that there does not exist in Euclidean 3-space any smooth surface whose geodesic geometry represents the entire hyperbolic plane. Nevertheless, it was Beltrami's pseudosphere which, more than anything else, convinced mathematicians that Lobachevsky's hyperbolic geometry was as consistent as Euclid's (page 225, Faber).

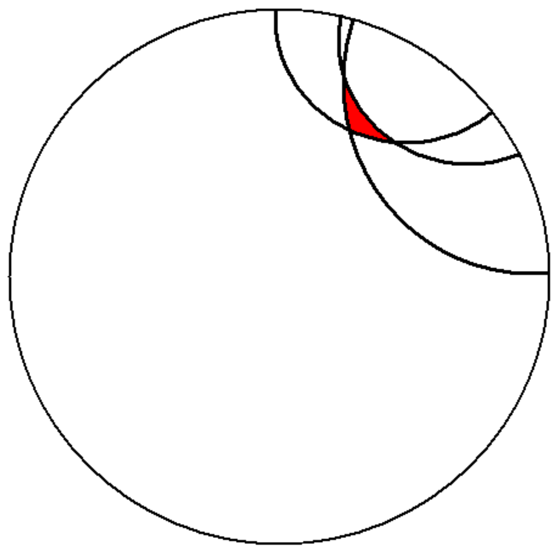
Note. Let's step aside and discuss curvature of surfaces briefly. Crudely, a surface has zero curvature if it is flat. For a smooth surface, we can describe curvature at a point P by considering tangent planes to the surface at point P. If a tangent plane to a surface at point P has the surface lying entirely on one side of the plane in a deleted neighborhood of P, then the surface has positive curvature at point P. For example, a sphere has positive (constant) curvature at each point. If a tangent plane to a surface at point P has part of the surface on one side of the plane and part on the other side for any deleted neighborhood of P, then the surface has negative curvature. For example, the saddle surface in 3-dimensional Euclidean space with formula $z = x^2 - y^2$ has negative curvature at each point. However, curvature is not a constant on the saddle surface! Curvature is its most extreme at the saddle point $(0, 0, 0)$ where it is -1, and is less extreme (closer to 0) away from the saddle point (where it is closer to a flat surface). THIS is why the saddle surface cannot be used for a model of hyperbolic geometry!!! It is not a surface of constant curvature, and if, for example, we take a triangle and slide it around the surface (translate it), then its angle sum will change while the length of the edges remain the same (we don't have "translation invariance" in this sense and are lead to contradictions involving congruent hyperbolic triangles and areas).

Problem. Describe the curvature at various points of a torus.

Definition. The *Poincare disk* model of hyperbolic geometry represents the "plane" as an open unit disk, "points" of the plane are points of the disk, and "lines" are circular arcs which are perpendicular to the boundary of the disk.



Note. We can use the model to illustrate some of the results for hyperbolic geometry. The above figure has two lines through point P both of which are parallel to the other line l, thus illustration the negation of Playfair's Theorem. To see that the sum of the angles of a triangle are less than $180°$, consider the following:

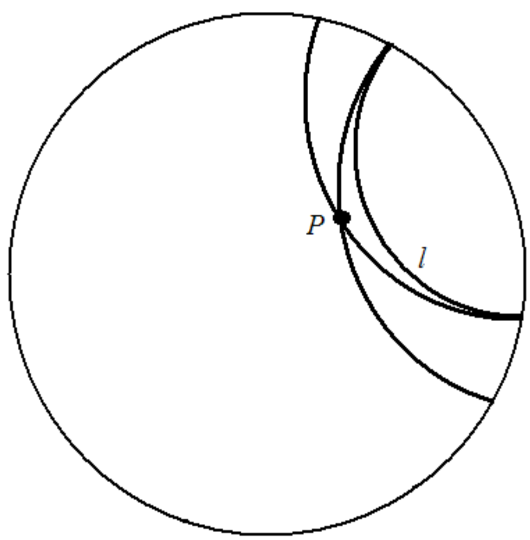


Note. The obvious shortcoming of the model is that line segments are apparently not infinite in extent (it looks as if the longest line in this universe is 2 units). However, we modify the way distance is measured. We describe the disk as $\{(x, y) \mid x^2+y^2 < 1\}$ in the Cartesian plane. We then take the differential of arclength, ds, as satisfying $ds^2 = \dfrac{dx^2 + dy^2}{1 - (x^2 + y^2)}$. Now if we consider the diameter D of the disk from $(-1, 0)$ to $(1, 0)$, we have the length

$$\int_D ds = \int_{-1}^{1} \frac{dx}{1 - x^2} = \frac{1}{2} \ln \left| \frac{1 + x}{1 - x} \right| \Bigg|_{-1}^{1} = \infty.$$

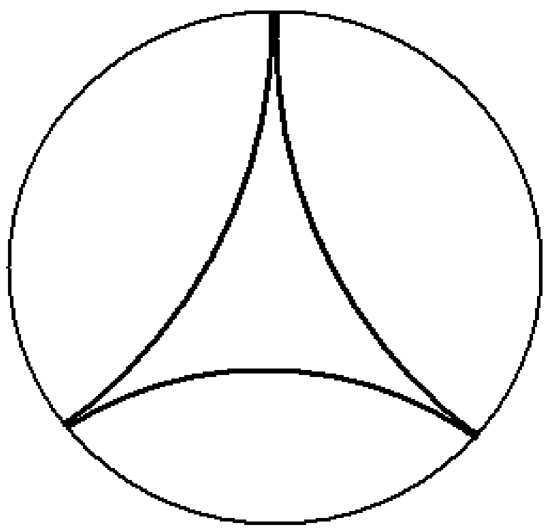
In general, "lines" in the Poincare disk are infinite (under this measure of arclength). In terms of curvature, the Poincare disk has curvature -4 at each point (to get curvature -1, we take a disk of radius 4).

Note. (From *A Survey of Classical and Modern Geometries* by Arthur Baragar.) Let's consider the negation of Playfair's Theorem again. Let L by a line and P a point not on l. Then there are two lines through P parallel to l and, hence, an infinite number of such lines. We might imagine tilting the lines through P towards two "limiting lines." If we tilt more than the limiting lines, we no longer have parallels to l. This is illustrated in the Poincare disk as follows:



The limiting lines are parallel to l and the other lines are said to be *ultraparallel* to l. With this verbiage, we have the following properties: (1) two lines which are ultraparallel have a common perpendicular, and (2) two lines which are parallel (but not ultraparallel) do not have a common perpendicular.

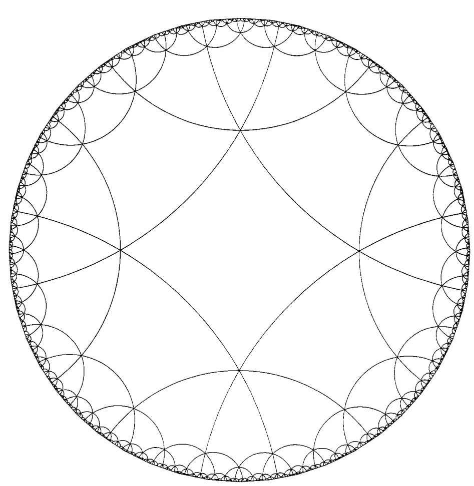
Note. We have said that the angle-sum of a hyperbolic triangle can be any value (strictly) between $0°$ and $180°$. By making the triangles small, the angle-sum approaches $180°$. We can make the angle-sum near $0°$ by making the triangles "large." In fact, we can consider "triply asymptotic triangles" with angle-sum $0°$:



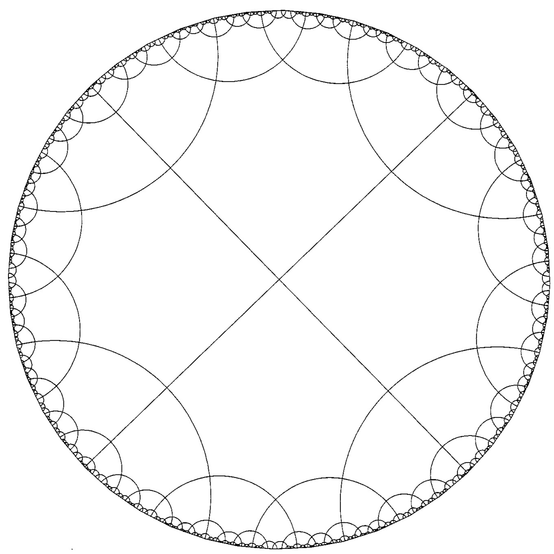
By previous results, we know that all triply asymptotic triangles are congruent, that the defect is $180°$, and that the area of each is maximal (since the defect is minimal).

Note. Clearly we can tile (or "tessellate") the Euclidean plane with squares. Notice that at each "vertex," four squares meet and the four right angles sum to $360°$. We can also tile the Euclidean plane with regular hexagons such that three hexagons meet at each vertex (the interior angle of a hexagon is $120°$). The hexagons can be subdivided into six triangles each to give an equilateral triangle tiling of the plane with six triangles meeting at each vertex. Each of these is an example of a regular polygon tiling of the Euclidean plane (recall that the interior angles of a regular Euclidean n-gon are each $\frac{n-2}{n} \times 180°$; this, combined with the fact that the angles of the n-gon must divide $360°$, implies that the above three tilings are the *only* regular n-gon tilings of the Euclidean plane).

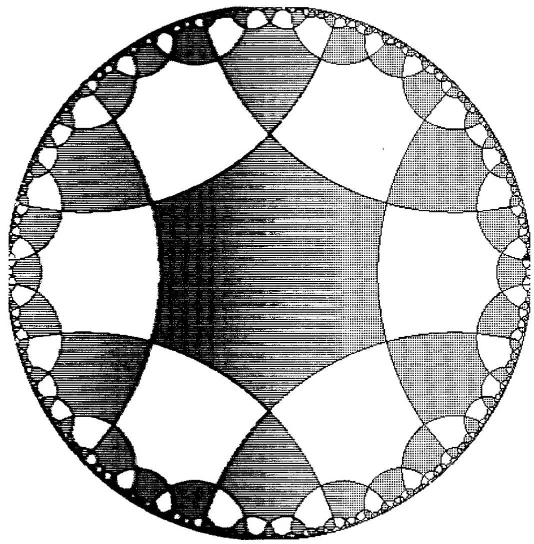
Note. In hyperbolic geometry, we define a polygon as *regular* if the lengths of the sides are the same and the angles are equal. The sizes of the angles, however, depend on the lengths of the edges. We can construct a regular n-gon whose angles are equal to α for any α such that $0 < \alpha < \frac{n-2}{n} \times 180°$. This means that we cannot take the hyperbolic plane with regular 4-gons where four meet at each vertex (traditional Euclidean right-angle squares do not exist here), but we can tile the plane with regular 4-gons where five (or any integer greater than 4) of them meet at each vertex. Here is an image of such a tiling with six 4-gons meeting at each vertex (from Baragar, pages 183):



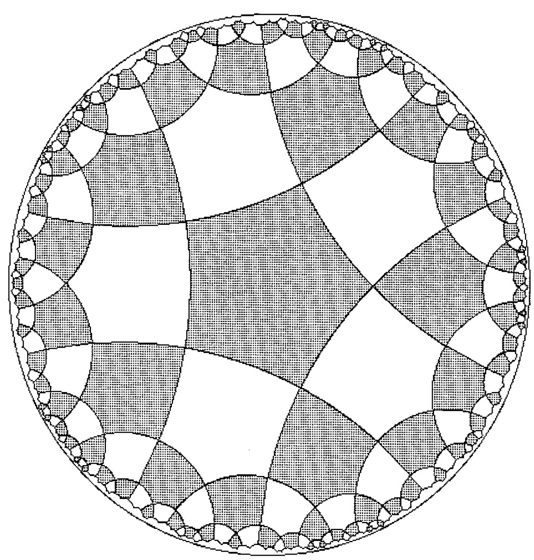
Notice that the angle-sum of each 4-gon in these figures is $240°$, so the area is $k240°$ for some constant k (determined by the curvature of the plane). Because of the liberty in the sizes of angles of regular n-gons in the hyperbolic plane, regular tilings are much more prevalent than in the Euclidean plane. Here's a hexagon tiling with four hexagons meeting at each vertex (Baragar, page 184):



In fact, the last two figures give tilings which are *duals* of each other. A dual of a tiling is determined by putting a vertex in the center of each n-gon and then joining vertices which are in n-gons that share an edge. Here is another view of the same tiling (*Geometry: Plane and Fancy*, David Singer, page 60):

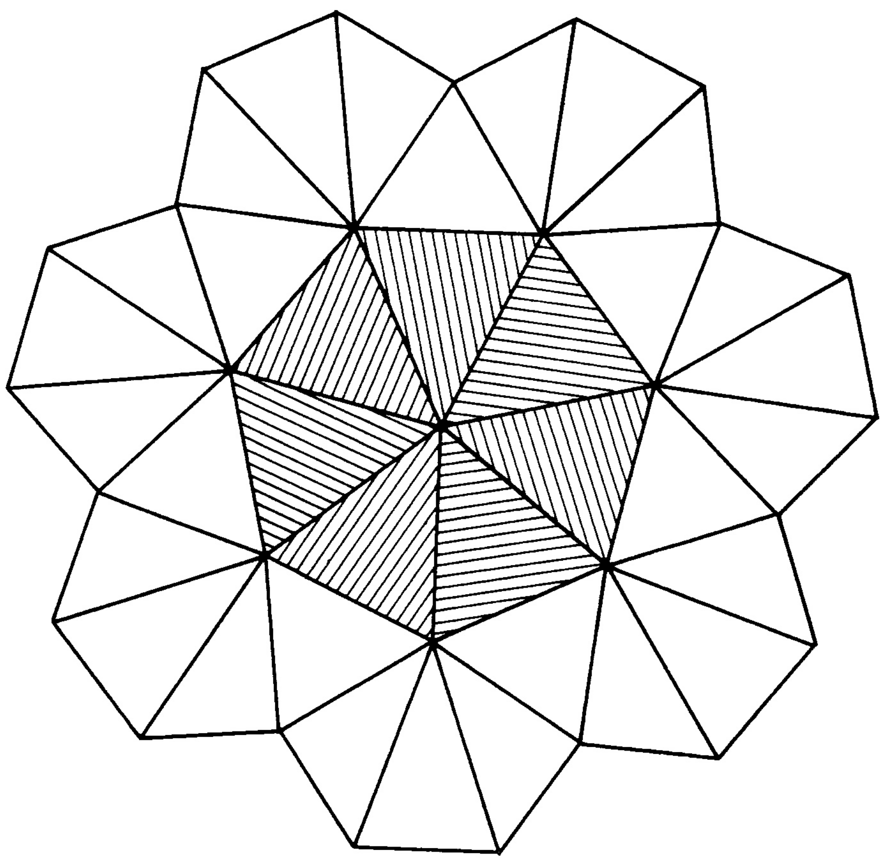


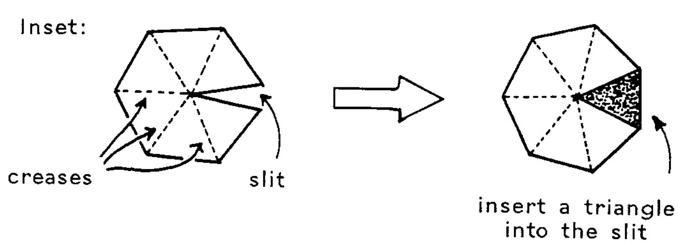
Here's a regular 5-gon tiling with four 5-gons meeting at a vertex (what's the dual?, from Singer page 61):



Some of these tilings are used in the paintings of M.C. Escher (1898–1972).

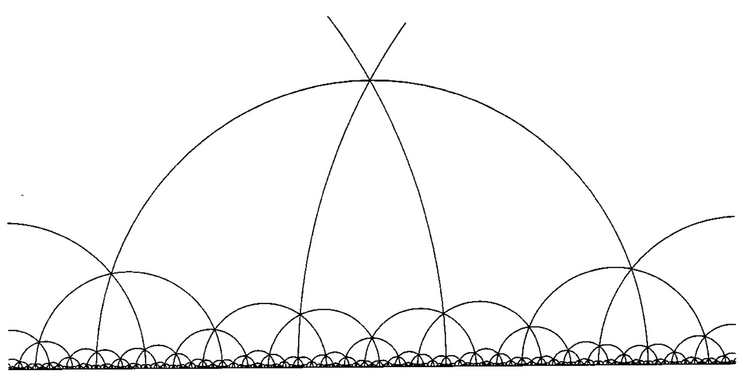
Note. With this tiling stuff, we see a way to construct a hyperbolic surface which we can hold in our hands. The easiest way is to take a bunch of equilateral triangles and tape them together with seven triangles joined at each vertex. Of course, you have to bend the paper for this to happen. The bending and twisting yields a surface of negative curvature (from Weeks, page 153):



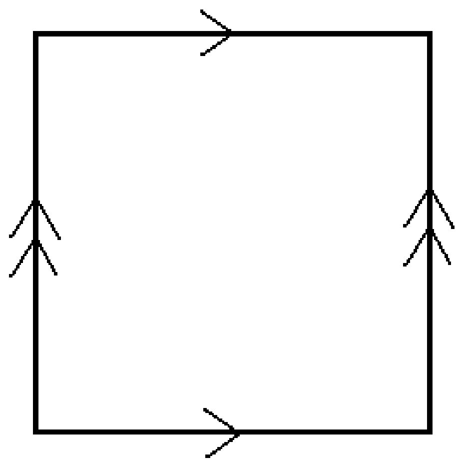


Some interesting "constructions" of surfaces of negative curvature (including the pseudosphere) using crocheting are given in *Experiencing Geometry in Euclidean, Spherical, and Hyperbolic Geometries*, 2nd Edition, by David Henderson.

Note. As another example of a model of hyperbolic geometry, we consider the *Poincare upper half plane* model. We take as the hyperbolic plane the upper half of the Cartesian plane, $\{(x, y) \mid y > 0\}$, and the hyperbolic lines are semicircles with centers on the x-axis and vertical (Cartesian) lines. Lines are infinite and the differential of arclength is $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$. The curvature of this surface is -1. It can be shown that all models of hyperbolic geometry are the same ("isomorphic"). Here is an image of a regular 4-gon tiling of the Poincare upper half plane with six 4-gons meeting at each vertex (from Baragar, pages 182):

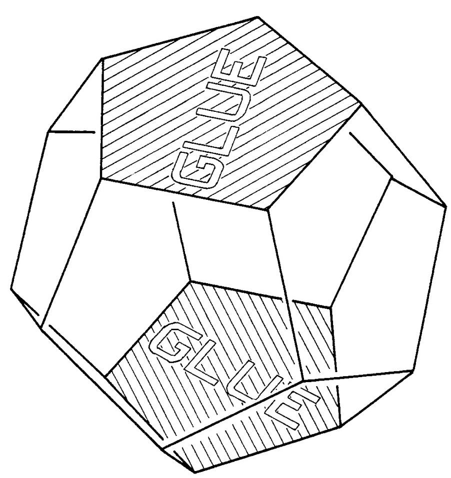


Note. As a final comment, we might wonder about the geometry of our universe. Let's start the discussion by considering the surfaces which are bounded but finite. We can create a torus by starting with a square and then "conceptually connecting" (versus a physical bending and connecting) opposite sides:



If we travel around this 2-D universe and pass through the right hand side, then we return on the left hand side, and similarly for the top and bottom. The square is called the *fundamental domain* and if we look around this universe, we see the "sky" tiled with copies of the fundamental domain. This is called a 2-torus (this is the universe PacMan lives in!). In fact (as I understand it) we can use the tiling and "conceptual connections" to create other interesting surfaces. For example, we could take a tiling of the hyperbolic plane with squares and take a tiling of the hyperbolic plane with squares and make the same connections as above. This creates a two dimensional surface (its global topology is the same as the flat 2-torus) with negative curvature. Notice that the curvature does not depend on how the surface is embedded in a higher dimensional space, but is simply an artifact of how distances are measured and what geodesics (lines) are. That is, the curvature is an *intrinsic property* of the surface. When you hear someone describe curvature as how a surface (or higher dimensional manifold) bends into a higher dimensional space, this is not correct! The curvature can be present without the appeal of a bigger space (this makes the discussion of "hyperspace" rather subtle!).

Note. So what about 3-dimensional spaces? Well, we could tile Euclidean 3-space with cubes and conceptually connect opposite faces. This produces the 3-torus. It has Euclidean geometry and a "nontrivial global topology" (there are directions in which you can travel which bring you back to where you started). It is possible to tile 3-D hyperbolic space with dodecahedra. If opposite faces of these dodecahedra are conceptually joined (this requires a small twist of the dodecahedron), this produces a 3-D finite space of negative curvature called the Seifert-Weber manifold:



This space has negative curvature, but is finite in extent. You may hear it said (by astronomers maybe) that if space has negative curvature then it must "bend away" from itself and so be infinite in extent. This is not true! It is true that if a space has positive curvature (and hence elliptic geometry) then it must curve back on itself and *must* be finite. So, for the universe, we have the following possibilities in terms of curvature and boundedness:

1. Zero curvature and infinite (for example, $\mathbb{R}^3$).
2. Zero curvature and finite (for example, 3-torus).
3. Negative curvature and infinite (for example, hyperbolic 3-space).
4. Negative curvature and finite (for example, the Seifert Weber space).
5. Positive curvature and finite (for example, the 3-sphere).

Studies are currently underway to try to determine the curvature and global topology of the universe. For a nice, easy to read account of these ideas, see *The Shape of Space*, 2nd edition, Jeffrey Weeks. A much more technical account is in *Three-Dimensional Geometry and Topology*, Volume 1, William Thurston, edited by Silvio Levy.