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An Eneström-Kakeya theorem for new classes of polynomials. (English summary)

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This paper considers conditions on the coefficients of a polynomial P in one complex variable z that allow one to find an annulus that contains all the zeros of P. These results are part of a growing body of literature that consists of generalizations of the Eneström-Kakeya Theorem, which states that if $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$, then

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \tag{1}$$

has all its zeros in $\{z\colon |z| \leq 1\}$. The most specific motivation for this paper comes from [J. Cao and R. B. Gardner, J. Comput. Appl. Math. **155** (2003), no. 1, 153–162; MR1992295], which provides an explicit annulus containing all the zeros of polynomials $P(z)$ of the form (1), whose coefficients satisfy (with $\alpha_j = \operatorname{Re} a_j$ and $\beta_j = \operatorname{Im} a_j$) the following conditions:

$$\alpha_0 \leq t^2\alpha_2 \leq t^4\alpha_4 \leq \cdots \leq t^{2k}\alpha_{2k} \geq t^{2k+2}\alpha_{2k+2} \geq \cdots \geq t^{2\lfloor n/2\rfloor}\alpha_{2\lfloor n/2\rfloor},$$
$$\alpha_1 \leq t^2\alpha_3 \leq t^4\alpha_5 \leq \cdots \leq t^{2\ell-2}\alpha_{2\ell-1} \geq t^{2\ell}\alpha_{2\ell+1} \geq \cdots \geq t^{2\lfloor n/2\rfloor}\alpha_{2\lfloor (n+1)/2\rfloor-1},$$
$$\beta_0 \leq t^2\beta_2 \leq t^4\beta_4 \leq \cdots \leq t^{2s}\beta_{2s} \geq t^{2s+2}\beta_{2s+2} \geq \cdots \geq t^{2\lfloor n/2\rfloor}\beta_{2\lfloor n/2\rfloor},$$
$$\beta_1 \leq t^2\beta_3 \leq t^4\beta_5 \leq \cdots \leq t^{2q-2}\beta_{2q-1} \geq t^{2q}\beta_{2q+1} \geq \cdots \geq t^{2\lfloor n/2\rfloor}\beta_{2\lfloor (n+1)/2\rfloor-1},$$

for some $t > 0$ and $k, \ell, s, q \in \{0, 1, \ldots, \lfloor n/2\rfloor\}$ (see Theorem 1.4). Notice that the hypotheses on the coefficients of P are different for the coefficients of even and odd powers of z.

The main result of this paper (Theorem 2.1) is an extension of the aforementioned result that considers comparable hypotheses on the coefficients of P, but with a separate condition for the coefficients of powers of z in each congruence class modulo m (for some $m < n$). The conclusion is the same in that it provides a description of an annulus that must contain all the zeros of P.

The main tool used in the proof of the main result is a result from E. C. Titchmarsh's book [*The theory of functions*, second edition, Oxford Univ. Press, Oxford, 1939; MR3728294], which bounds the number of zeros of an analytic function in a disk in terms of its maximum modulus on a slightly larger disk. A similar method was recently used to prove related results in [A. Mir, A. Ahmad and A. H. Malik, J. Math. Appl. **42** (2019), 135–146; MR3992912].

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