SOME INEQUALITIES FOR THE MAXIMUM MODULUS OF RATIONAL FUNCTIONS

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Abstract. For a polynomial \( p(z) \) of degree \( n \), it follows from the Maximum Modulus Theorem that
\[
\max_{|z|=1} |p(z)| \leq R^n \max_{|z|=1} |p(z)|.
\]
It was shown by Ankeny and Rivlin in 1955 that if \( p(z) \neq 0 \) for \( |z| < 1 \) then
\[
\max_{|z|=1} |p(z)| \leq R^n + 1 \max_{|z|=1} |p(z)|.
\]
These two results were extended to rational functions by Govil and Mohapatra [4]. In this paper, we give refinements of these results of Govil and Mohapatra.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let \( \mathcal{P}_n \) denote the set of all complex algebraic polynomials \( p \) of degree at most \( n \) and let \( p' \) be the derivative of \( p \). For a function \( f \) defined on the unit circle \( T = \{ z \mid |z| = 1 \} \) in the complex plane \( \mathbb{C} \), set \( \|f\| = \sup_{z \in T} |f(z)| \), the Chebyshev norm of \( f \) on \( T \).

Let \( \mathbb{D}_- \) denote the region strictly inside \( T \), and \( \mathbb{D}_+ \) the region strictly outside \( T \). For \( a_v \in \mathbb{C}, v = 1, 2, \ldots, n \), let \( w(z) = \prod_{v=1}^{n} (z-a_v) \), \( B(z) = \prod_{v=1}^{n} (1-\overline{a_v} z)/(z-a_v) \) be the Blashke product, and \( \mathcal{R}_n = \mathcal{R}_n(a_1, a_2, \ldots, a_n) = \{ p(z)/w(z) \mid p \in \mathcal{P}_n \} \).

Then \( \mathcal{R}_n \) is the set of rational functions with possible poles at \( a_1, a_2, \ldots, a_n \) and having a finite limit at \( \infty \). Also note that \( B(z) \in \mathcal{R}_n \).

DEFINITIONS.

i: For polynomial \( p(z) = \sum_{v=0}^{n} a_v z^v \), the conjugate transpose (reciprocal) \( p^* \) of \( p \) is defined by
\[
p^*(z) = \overline{p(1/z)} = \overline{z^n p(1/z)} = \overline{a_0} z^n + \overline{a_1} z^{n-1} + \cdots + \overline{a_n}.
\]

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For rational function \( r(z) = p(z)/w(z) \in \mathcal{R}_n \), the conjugate transpose, \( r^* \), of \( r \) is defined by
\[
r^*(z) = B(z)\overline{r(1/\overline{z})} = B(z)\overline{r(1/z)}.
\]

The polynomial \( p \in \mathcal{P}_n \) is self-inversive if \( p^*(z) = \lambda p(z) \) for some \( \lambda \in \mathbb{T} \).

The rational function \( r \in \mathcal{R}_n \) is self-inversive if \( r^*(z) = \lambda r(z) \) for some \( \lambda \in \mathbb{T} \).

It is easy to verify that if \( r \in \mathcal{R}_n \) and \( r = p/w \), then \( r^* = p^*/w \) and hence \( r^* \in \mathcal{R}_n \).

So \( p/w \) is self-inversive if and only if \( p \) is self-inversive.

Govil and Mohapatra [4] gave a result analogous to inequality (1), but for rational functions, as follows.

**THEOREM A.** If
\[
r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^n (z - a_v)} \in \mathcal{R}_n
\]
is a rational function with \( |a_v| > 1 \) for \( 1 \leq v \leq n \), then for \( |z| \geq 1 \),
\[
|r(z)| \leq \|r\| |B(z)|.
\]

This result is best possible and equality holds for \( r(z) = \lambda \prod_{v=1}^n \frac{1 - \overline{a_v}z}{z - a_v} = \lambda B(z) \) where \( \lambda \in \mathbb{C} \).

In the same paper, Govil and Mohapatra [4] also proved a result given below that is analogous to inequality (2), but is for rational functions, as follows.

**THEOREM B.** Let
\[
r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^n (z - a_v)} \in \mathcal{R}_n
\]
with \( |a_v| > 1 \) for \( 1 \leq v \leq n \). If all the zeros of \( r \) lie in \( \mathbb{T} \cup \mathbb{D}_+ \), then for \( |z| \geq 1 \)
\[
|r(z)| \leq \|r\| \frac{|B(z)| + 1}{2}.
\]

This result is best possible and equality holds for the rational function \( r(z) = \alpha B(z) + \beta \) where \( |\alpha| = |\beta| \).

In this paper we prove the following refinements of the above two theorems. Here \( p(z) = \sum_{v=0}^n \alpha_v z^v \) is a polynomial of degree \( n \).
THEOREM 1.1. If
\[ r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^{n} (z - a_v)} \in \mathcal{R}_n \]
is a rational function with \(|a_v| > 1\), \(1 \leq v \leq n\), then for \(|z| \geq 1\),
\[ |r(z)| \leq ||r|| \cdot |B(z)| \left\{ 1 - \frac{(||r - |r^*(0)||)(|z| - 1)}{|r^*(0)| + |z| ||r||} \right\}. \tag{5} \]
The result is best possible and equality holds for \(r(z) = \lambda B(z)\) where \(\lambda \in \mathbb{C}\).

It is clear that Theorem 1 sharpens Theorem A. Also, we can use Theorem 1 to derive a sharpening form of Bernstein’s Inequality for polynomials. For this, let \(p(z) = \sum_{v=0}^{n} \alpha_v z^v\) be a polynomial of degree \(n\). Then
\[ r(z) = \frac{p(z)}{\prod_{v=1}^{n} (z - a_v)} \in \mathcal{R}_n \]
and hence by Theorem 1, for \(|z| \geq 1\),
\[ \left| \frac{r(z)}{B(z)} \right| = \left| \frac{p(z)}{\prod_{v=1}^{n} (1 - \overline{a_v}z)} \right| \leq ||r|| \left\{ 1 - \frac{(||r - |r^*(0)||)(|z| - 1)}{|r^*(0)| + |z| ||r||} \right\}. \tag{6} \]
If \(z^*\) on \(|z| = 1\) is such that
\[ ||r|| = |r(z^*)| = \frac{|p(z^*)|}{\prod_{v=1}^{n} (z^* - a_v)} \]
then we get from (6)
\[ \left| \frac{p(z)}{\prod_{v=1}^{n} (1 - \overline{a_v}z)} \right| \leq \frac{|p(z^*)|}{\prod_{v=1}^{n} |z^* - a_v|} \left\{ 1 - \frac{(||p - |p^*(0)||)(|z| - 1)}{|p^*(0)| \prod_{v=1}^{n} |z^* - a_v| + |z| |p(z^*)|} \right\}. \tag{8} \]
Since \(p(z) = \sum_{v=0}^{n} \alpha_v z^v\) and \(r^*(z) = \frac{p^*(z)}{\prod_{v=1}^{n} (z - a_v)}\), we get \(|r^*(0)| = \frac{|\alpha_n|}{\prod_{v=1}^{n} |a_v|}\) and therefore from (8) we have for \(|z| > 1\),
\[ |p(z)| \leq |p(z^*)| \prod_{v=1}^{n} \left| \frac{1 - \overline{a_v}z}{z^* - a_v} \right| \left\{ 1 - \frac{|p(z^*)| - |\alpha_n| \prod_{v=1}^{n} |(z^* - a_v)/a_v|)(|z| - 1)}{|\alpha_n| \prod_{v=1}^{n} |(z^* - a_v)/a_v| + |z| |p(z^*)|} \right\}. \tag{9} \]
Since (9) holds for all \(|a_v| \geq 1\), where \(1 \leq v \leq n\), making \(|a_v| \to \infty\), where \(1 \leq v \leq n\), we get that for \(|z| \geq 1\),
\[ |p(z)| \leq |p(z^*)| \prod_{v=1}^{n} \left| \frac{1 - \overline{a_v}z}{z^* - a_v} \right| \left\{ 1 - \frac{|p(z^*)| - |\alpha_n|)(|z| - 1)}{|\alpha_n| + |z| |p(z^*)|} \right\}. \tag{10} \]
We show in Lemma 5 in the next section that (10) implies for \(|z| \geq 1\)
\[ |p(z)| \leq ||p|| \cdot |z|^n \left\{ 1 - \frac{(||p - |\alpha_n||)(|z| - 1)}{|\alpha_n| + |z| ||p||} \right\}, \]
which is equivalent to that for \(|z| = R \geq 1\),
\[ |p(z)| \leq R^n \left\{ 1 - \frac{(||p - |\alpha_n||)(R - 1)}{|\alpha_n| + R ||p||} \right\} ||p||. \]
This rate of growth result for a polynomial, which is a sharpening of Bernstein Inequality, first appeared as Lemma 3 of [2].

As a refinement of Theorem B, we shall prove...
THEOREM 1.2. Let
\[ r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^{n}(z - a_v)} \in R_n \]
with \(|a_v| > 1 \) for \( 1 \leq v \leq n \). If all the zeros of \( r \) lie in \( \mathbb{T} \cup \mathbb{D}_+ \), then for \(|z| \geq 1\)
\[ |r(z)| \leq \frac{1}{2} \left( |r||\{B(z)| + 1\} - (|B(z)| - 1) \min_{|z|=1} |r(z)| \right). \]

Clearly Theorems 1.1 and 1.2 without any additional hypotheses, give bounds that are sharper than those obtainable from Theorems A and B respectively.

2. LEMMAS

The following is a well known generalization of Schwarz’s Lemma (see, for example, [3]).

LEMMA 2.1. If \( f \) is analytic inside and on the circle \(|z| = 1\), then for \(|z| \leq 1\),
\[ |f(z)| \leq \|f\| |z| + |f(0)|. \] (11)
The next two results are due to Govil and Mohapatra [4].

LEMMA 2.2. Let \( r \in R_n \) with all its poles in \( \mathbb{D}_+ \). If \( r \) has all its zeros in \( \mathbb{T} \cup \mathbb{D}_+ \), then for all \(|z| \geq 1\), \(|r(z)| \leq |r^*(z)|\).

LEMMA 2.3. Let \( r \in R_n \) with all its poles in \( \mathbb{D}_+ \). Then for \(|z| \geq 1\),
\[ |r(z)| + |r^*(z)| \leq \|r\|(|B(z)| + 1). \]

LEMMA 2.4. Let \( r \in R_n \) with all its poles in \( \mathbb{D}_+ \). If \( r \) has all its zeros in \( \mathbb{T} \cup \mathbb{D}_+ \), then for \(|z| \geq 1\), we have
\[ |r(z)| + (|B(z)| - 1) \min_{|z|=1} |r(z)| \leq |r^*(z)|. \] (12)

Proof. Since the rational function \( r \) has no zeros in \( \mathbb{D}_- \) hence for every \( \alpha \in \mathbb{C} \) with \(|\alpha| < 1\), the rational function \( r(z) - \alpha \min_{|z|=1} |r(z)| \) has no zero in \( \mathbb{D}_- \) and has all its poles, like \( r \), in \( \mathbb{D}_+ \). Applying Lemma 2.2 to \( r(z) - \alpha \min_{|z|=1} |r(z)| \) we get that for \(|z| \geq 1\)
\[ |r(z)| + (|B(z)| - 1) \min_{|z|=1} |r(z)| \leq |r^*(z)|, \]
and so for \(|z| \geq 1\),
\[ |r(z)| - |\alpha| \min_{|z|=1} |r(z)| \leq |r^*(z) - B(z)\alpha \min_{|z|=1} |r(z)||, \]
With the appropriate choice of arg(\( \alpha \)) we then have for \(|z| \geq 1\),
\[ |r(z)| - |\alpha| \min_{|z|=1} |r(z)| \leq |r^*(z)| - |\alpha||B(z)| \min_{|z|=1} |r(z)|. \] (13)
Note that \( r \) has no zeros in \( \mathbb{D}_- \) and so is analytic in \( |z| \leq 1 \). Hence by the Minimum Modulus Theorem, we have \(|r(z)| > |\alpha| \min_{|z|=1} |r(z)| \) for \(|z| \leq 1 \). Therefore for \(|z| \geq 1 \) we get
\[
|r^*(z)| = |B(z)\overline{r(1/z)}| = |B(z)| |r(1/z)| > |\alpha| |B(z)| \min_{|z|=1} |r(z)|,
\]
which clearly implies that the right-hand side of (13) is positive. Making \( |\alpha| \to 1 \) in (13), we easily get
\[
|r(z)| + (|B(z)| - 1) \min_{|z|=1} |r(z)| \leq |r^*(z)|, \quad \text{for } |z| \geq 1,
\]
which is (12), and thus the proof of Lemma 2.4 is complete. \( \square \)

**Lemma 2.5.** The function
\[
g(x) = x \left\{ 1 - \frac{(x - |\alpha_n|)(|z| - 1)}{|\alpha_n| + |z|x} \right\},
\]
where \( \alpha_n, z \in \mathbb{C} \) with \( z \neq 0 \), is an increasing function for \( x \geq 0 \).

**Proof.** We have
\[
g'(x) = \frac{|z|x^2 + 2|\alpha_n|x + |z||\alpha_n|^2}{(|\alpha_n| + |z|x)^2} \geq 0
\]
for \( x \geq 0 \). So \( g \) is an increasing function for \( x \geq 0 \), as claimed. \( \square \)

3. Proofs of Theorems

**Proof of Theorem 1.1.** Since
\[
r(z) = \frac{p(z)}{w(z)} = \frac{p(z)}{\prod_{v=1}^{n}(z - a_v)} \in \mathcal{R}_n
\]
with \( |a_v| > 1 \) for \( 1 \leq v \leq n \), the function \( r^*(z) = p^*(z)/\prod_{v=1}^{n}(z - a_v) \) is analytic in \( |z| \leq 1 \). Therefore by Lemma 2.1 we get that, for \( |z| \leq 1 \),
\[
|r^*(z)| \leq \|r^*\| \frac{\|r\||z| + |r^*(0)|}{|r^*(0)||z| + \|r\|}
\]
and since \( \|r^*\| = \|r\| \), inequality (14) is in fact equivalent to the inequality that, for \(|z| \leq 1 \),
\[
|r^*(z)| \leq \|r\| \frac{\|r\||z| + |r^*(0)|}{|r^*(0)||z| + \|r\|}
\]
Since by definition \( r^*(z) = B(z)\overline{r(1/z)} \), we get from (15) that for \(|z| \leq 1 \),
\[
\overline{r(1/z)} \leq \frac{\|r\|}{|B(z)|} \frac{\|r\||z| + |r^*(0)|}{|r^*(0)||z| + \|r\|}
\]
which clearly gives that for \(|z| \geq 1 \),
\[
|r(z)| \leq \frac{\|r\|}{|B(1/z)|} \frac{\|r\||z| + |r^*(0)||z|}{|r^*(0)| + \|r\||z|}
\]
It is clear from the definition of $B(z)$ that $|B(1/\tau)| = 1/|B(z)|$ and this, when combined with (16), gives that for $|z| \geq 1$,

$$|r(z)| \leq \|r\||B(z)||r| + |r^*(0)||z| \over |r^*(0)| + \|r\||z|$$

$$= \|r\||B(z)|(1 - ((|r| - |r^*(0)|)(|z| - 1)) \over |r^*(0)| + \|r\||z|),$$

which is (5) and this completes the proof of the Theorem 1.1. \Box

**PROOF OF THEOREM 1.2.** Since $r \in \mathcal{R}_n$ and has all its poles in $\mathbb{D}_+$ hence, by Lemma 2.3, for $|z| \geq 1$ we have

$$(17) \quad |r(z)| + |r^*(z)| \leq \|r\||(B(z)| + 1).$$

Because $r$ has all its zeros in $\mathbb{T} \cup \mathbb{D}_+$ therefore we can apply Lemma 2.4 to $r$, and this will give that for $|z| \geq 1$,

$$(18) \quad |r(z)| + (|B(z)| - 1) \min_{|z|=1} |r(z)| \leq |r^*(z)|.$$

Combining the conclusion of (18) with (17) we get that for $|z| \geq 1$.

$$2|r(z)| + (|B(z)| - 1) \min_{|z|=1} |r(z)| \leq \|r\||(B(z)| + 1),$$

which is clearly equivalent to

$$|r(z)| \leq \frac{1}{2} \left( \|r\||(B(z)| + 1) - (|B(z)| - 1) \min_{|z|=1} |r(z)| \right),$$

and the proof of Theorem 1.2 is thus complete. \Box

**References**