3-TRANSROTATIONAL STEINER TRIPLE SYSTEMS

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Abstract. A Steiner triple system of order v, denoted $STS(v)$, is said to be k-transrotational if it admits an automorphism consisting of a fixed point, a transposition, and k cycles of length $\frac{v-3}{k}$. We show that a 3-transrotational $STS(v)$ exists if and only if $v \equiv 3$, 9 or 15 (mod 24).

1. Introduction

A *Steiner triple system* of order v, denoted $STS(v)$, is a v-element set, X, of points, together with a set β, of unordered triples of elements of X, called *blocks*, such that any two points of X are together in exactly one block of β. It is well known that a $STS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6). An *automorphism* of a $STS(v)$ is a permutation π of X which fixes β. A permutation π of a v-element set is said to be of *type* $[\pi] = [p_1, p_2, ..., p_v]$ if the disjoint cyclic decomposition of π contains p_i cycles of length i. So we have $\sum ip_i = v$. The *orbit* of a block under an automorphism, π, is the image of the block under the powers of π. A set of blocks, B, is said to be a *set of base blocks for a STS(v) under the permutation* π if the orbits of the blocks of B produce the $STS(v)$ and exactly one block of B occurs in each orbit.

The question of "For which orders v is there a $STS(v)$ admitting π as an automorphism?" has been explored for several types of automorphisms (for surveys of the results, see [1] and [4]). A *cyclic* $STS(v)$ is one admitting an automorphism of type $[0, 0, ..., 1]$ and exists if and only if $v \equiv 1$ or 3 (mod 6) and $v \neq 9$ [6]. A *reverse* $STS(v)$ admitting an automorphism of type $[1, \frac{v-1}{2}, 0, ..., 0]$ exists if and only if $v \equiv 1$, 3, 9, or 19 (mod 24) ([4], [8], and [9]). A *k-rotational* $STS(v)$ is one admitting an automorphism of type $[1, 0, ..., 0, k, 0, ..., 0]$. A 1-rotational $STS(v)$ exists if and only if $v \equiv 3$ or 9 (mod 24), a 2-rotational $STS(v)$ exists if and only if $v \equiv 1$, 3, 7, 9, 15, or 19 (mod 24), and a 6-rotational $STS(v)$ exists if and only if $v \equiv 1$, 7, or 19 (mod 24) [7]. A 3-rotational $STS(v)$ exists if and only if $v \equiv 1$ or 19 (mod 24) and a 4-rotational $STS(v)$ exists if and only if $v \equiv 1$ or 9 (mod 12) [2]. A $STS(v)$ admitting an automorphism of type $[1, 1, 0, ..., 0, k, 0, ..., 0]$ is said to be *k-transrotational.* A 1-transrotational $STS(v)$ exists if and only if $v \equiv 1$, 7, 9, or 15 (mod 24) and a 2-transrotational $STS(v)$ exists if and only if $v \equiv 3$ or 19 (mod 24) [5]. In this paper, we address necessary and sufficient conditions for the existence of 3-transrotational $STS(v)$.

2. The existence of 3-transrotational Steiner triple systems

A 3-transrotational $STS(v)$ admits an automorphism of the type $[1,1,0,\dots,0,3,0,\dots,0]$. We immediately have the following necessary condition.

Lemma 2.1 *A necessary condition for the existence of a 3-transrotational $STS(v)$ is that $v \equiv 3$, 9, or 15 (mod 24).*

Proof. Since the automorphism of such a system contains 3 cycles of length $\frac{v-3}{3}$, we need $3|(v-3)$. This implies that $v \equiv 3$, 9, 15 or 21 (mod 24). However, if we have a 3-transrotational $STS(v)$ where $v \equiv 21$ (mod 24), then the automorphism π has 3 cycles of length $N = \frac{v-3}{3}$ where $N/2$ is odd. But then $\pi^{N/2}$ is of type $[1, \frac{v-1}{2}, 0, \dots, 0]$ and the $STS(v)$ is also reverse. But a reverse $STS(v)$ does not exist for $v \equiv 21$ (mod 24). □

Sufficiency can be established as follows.

Lemma 2.2 *A 3-transrotational $STS(v)$ exists if $v \equiv 3$, 9 or 15 (mod 24)*

Proof. Let $v \equiv 9$ or 15 (mod 24). Then there exists a 1-transrotational $STS(v)$ admitting an automorphism π of type $[1,1,\dots,0,1,0,0,0]$. Then π^3 is of type $[1,1,\dots,0,3,0,\dots,0]$, and the $STS(v)$ is also 3-transrotational. Now suppose that $v \equiv 3$ (mod 24), say $v = 24k+3$. Let $N = \frac{v-3}{3}$. Then the following are base blocks from the set $\{\infty, a, b\} \bigcup Z_N \times \{1,2,3\}$ under the automorphism $(\infty)(a,b)(0_1, 1_1, \dots, (N-1)_1)(0_2, 1_2, \dots, (N-1)_2)(0_3, 1_3, \dots, (N-1)_3)$:

$(0_1, (4k-2-2r)_1, (4k-1-r)_2)$ for $r = 0, 1, \dots, 2k-2$,
$(0_2, (4k-2-2r)_2, (4k-1-r)_3)$ for $r = 0, 1, \dots, 2k-2$,
$(0_3, (4k-2-2r)_3, (4k-1-r)_1)$ for $r = 0, 1, \dots, 2k-2$,
$(0_1, (4k-1-2r)_1, (8k-1-r)_2)$ for $r = 0, 1, \dots, 2k-1$,
$(0_2, (4k-1-2r)_2, (8k-1-r)_3)$ for $r = 0, 1, \dots, 2k-1$,
$(0_3, (4k-1-2r)_3, (8k-1-r)_1)$ for $r = 0, 1, \dots, 2k-1$,
$(0_1, 0_2, 0_3)$, $(\infty, (2k)_1, 0_3)$, $(\infty, 0_2, (4k)_2)$,
(∞, a, b), $(a, 1_1, (4k+1)_1)$, $(a, 0_3, (4k)_3)$,
$(a, 0_1, (2k)_2)$, and $(a, 1_2, (2k+1)_3)$. □

We summerize the results in a theorem.

Theorem 2.3 *A 3-transrotational $STS(v)$ exists if and only if $v \equiv 3$, 9, or 15 (mod 24).*

REFERENCES

[1] R.S. Calahan, Automorphisms of Steiner triple systems with holes, Ph.D. Thesis, Auburn University, Auburn, AL (1990).
[2] C.J. Cho, Rotational Steiner triple systems, Discrete Math 42 (1982) 153-159.
[3] J. Doyen, A note on reverse Steiner triple systems, Discrete Math 1 (1972) 315-319.

[4] R. Gardner, Automorphisms of Steiner triple Systems, M.S. Thesis, Auburn University, Auburn, AL (1987).
[5] R. Gardner, Steiner triple systems with transrotational automorphisms, submitted.
[6] R. Peltesohn, Eine losung der beiden Heffterschen differenzenprobleme, Compositio Math. 6 (1939) 251-257.
[7] K.T. Phelps and A. Rosa, Steiner triple systems with rotational automorphisms, Discrete Math 33 (1981) 57-66.
[8] A. Rosa, On reverse Steiner triple systems, Discrete Math. 1 (1972) 61-71.
[9] L. Teirlinck, The existence of reverse Steiner triple systems, Discrete Math 6 (1973) 301-302.