DISCRETE MATHEMATICS

ELSEVIER

Discrete Mathematics 171 (1997) 121–139

Cyclic and rotational hybrid triple systems

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Abstract

A hybrid triple system of order v, denoted $HTS(v)$, is said to be *cyclic* if it admits an automorphism consisting of a single cycle of length v. A $HTS(v)$ admitting an automorphism consisting of a fixed point and a cycle of length $v-1$ is said to be *rotational*. Necessary and sufficient conditions are given for the existence of a cyclic $HTS(v)$ and a rotational $HTS(v)$.

1. Introduction

A *cyclic triple*, denoted $C(a, b, c)$, is the digraph on the vertex set $\{a, b, c\}$ with the arc set $\{(a, b), (b, c), (c, a)\}$. A *transitive triple*, denoted $T(a, b, c)$, is the digraph on the vertex set $\{a, b, c\}$ with the arc set $\{(a, b), (b, c), (a, c)\}$. Let D_v denote the complete symmetric digraph on v vertices and let $b_v = v(v-1)/3$. A *c-hybrid triple system* of order v, denoted c-$HTS(v)$, is an arc-disjoint partition of D_v into c cyclic triples and $t = b_v - c$ transitive triples.

A *directed triple system* of order v, denoted $DTS(v)$, is a 0-$HTS(v)$. A $DTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$ [7]. A *Mendelsohn triple system* of order v, denoted $MTS(v)$, is a b_v-$HTS(v)$. A $MTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$ [9]. In general, a c-$HTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$, and $c \in B_v = \{0, 1, 2, \ldots, b_v - 2, b_v\}$ [5, 6]. A related design is an *oriented triple system* (also called an *ordered triple system*) of order v, denoted $OTS(v)$, which is a c-$HTS(v)$ where c is any integer such that $0 \leqslant c \leqslant b_v$ (notice that c cannot equal $b_v - 1$). Therefore, an $OTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$ [8].

* Corresponding author.
[1] Supported by ETSU under RDC grant #94-105/M.
[2] Supported by MURST and GNSAGA.
[3] Supported by MTSU Faculty Research Committee grant #220715.

PII S0012-365X(96)00036-2

An *automorphism* of a c-$HTS(v)$ based on D_v is a permutation π of the vertex set of D_v which fixes the collection of triples of the c-$HTS(v)$. The *orbit* of a triple under an automorphism π is the collection of images of the triple under the powers of π. A set of triples β is a *set of base triples* for a c-$HTS(v)$ under the automorphism π if the orbits of the triples of β produce a set of triples for a c-$HTS(v)$ and exactly one triple of β occurs in each orbit.

A c-$HTS(v)$ admitting an automorphism consisting of a single cycle is said to be *cyclic*. A cyclic $DTS(v)$ exists if and only if $v \equiv 1, 4$, or $7 \pmod{12}$ [4] and a cyclic $MTS(v)$ exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$ [3]. A cyclic $OTS(v)$ exists if and only if $v \equiv 0, 1, 3, 4, 7$, or $9 \pmod{12}$, $v \neq 9$ [10].

A c-$HTS(v)$ admitting an automorphism consisting of a fixed point and a cycle of length $v - 1$ is said to be *rotational*. A rotational $DTS(v)$ exists if and only if $v \equiv 0 \pmod 3$ [2]. A rotational $MTS(v)$ exists if and only if $v \equiv 1, 3$, or $4 \pmod 6$, $v \neq 10$ [1]. A rotational $OTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$ [10].

The purpose of this paper is to present necessary and sufficient conditions for the existence of cyclic and rotational c-$HTS(v)$s.

2. Cyclic hybrid triple systems

In this section, we construct cyclic c-$HTS(v)$s based on the vertex set $\mathbf{Z}_v$ and admitting the automorphism $\pi = (0, 1, \ldots, v - 1)$. The orbit of any transitive triple under π is of length v. We therefore immediately have:

Lemma 2.1. *If a cyclic c-$HTS(v)$ exists, then $t = b_v - c \equiv 0 \pmod v$.*

Since a cyclic c-$HTS(v)$ is also an $OTS(v)$, we have [10].

Lemma 2.2. *If a cyclic c-$HTS(v)$ exists, then $v \equiv 0, 1, 3, 4, 7$, or $9 \pmod{12}$, $v \neq 9$.*

We give the remaining necessary conditions and establish sufficiency in the next five lemmas.

Lemma 2.3. *If $v \equiv 1 \pmod 6$ then there exists a cyclic c-$HTS(v)$ for all $c \in B_v$ such that $c \equiv 0 \pmod v$, except for the case $c = v(v - 4)/3$.*

Proof. We consider two cases based on the value of c.

Case 1: Suppose $v \equiv 1 \pmod 6$ and $c \equiv 0 \pmod{2v}$. Then there exists a cyclic Steiner triple system of order v [11]. Let B be a collection of base triples for such a system. Let $B = B_1 \cup B_2$ where $|B_1| = c/(2v)$. Define

$$H = \{C(a, b, c), C(c, b, a) \mid (a, b, c) \in B_1\} \cup \{T(a, b, c), T(c, b, a) \mid (a, b, c) \in B_2\}.$$

Then H is a set of base triples for a cyclic c-$HTS(v)$ where $c \in B_v$ and $c \equiv 0 \pmod{2v}$.

Case 2: Suppose $v \equiv 1 \pmod 6$, say $v = 6m + 1$, $m > 1$, and $c \equiv v \pmod{2v}$. If $c = v(v - 4)/3$ then the existence of a cyclic c-$HTS(v)$ is equivalent to partitioning the set $\{1, 2, \ldots, 6m\}$ into (difference) triples (x_i, y_i, z_i) for $i = 1, 2, \ldots, 2m - 1$ and (x, y, z) such that $x_i + y_i + z_i \equiv 0 \pmod{6m + 1}$ and $x + y \equiv z \pmod{6m + 1}$. Now,

$$\sum_{i=1}^{6m} i = \sum_{i=1}^{2m-1} (x_i + y_i + z_i) + (x + y + z) = 3m(6m + 1) \equiv 0 \pmod{6m + 1},$$

and so $x + y + z \equiv 0 \pmod{6m + 1}$. But $x + y + z \equiv 2c \pmod{6m + 1}$. Therefore $2z \equiv 0 \pmod{6m + 1}$, which is impossible.

Suppose then that $c = (2k + 1)\,v$ where $k = 0, 1, \ldots, m - 2$. Consider the set:

$$\{C(0, 5m + 1, 1), T(0, 5m, m), T(0, 1, 3m + 1), T(0, 3m + 2, 2m)\}$$

$$\cup \{C(0, m + 2 + i, 6m - 1 - i), C(0, 3m - 1 - i, 2 + i) \mid i = 0, 1, \ldots, k - 1$$
$$\text{(omit these triples if } k = 0 \text{ or if } m = 2)\}$$

$$\cup \{T(0, m + 2 + i, 6m - 1 - i), T(0, 3m - 1 - i, 2 + i) \mid i = k, k + 1, \ldots, m - 3$$
$$\text{(omit these triples if } k = m - 2 \text{ or if } m = 2)\}.$$

This is a set of base triples for a cyclic c-$HTS(v)$ under the permutation $\pi = (0, 1, \ldots, v - 1)$. □

Lemma 2.4. *If $v \equiv 4 \pmod{12}$, then there exists a cyclic c-$HTS(v)$ for all $c \in B_v$ such that $c \equiv 0 \pmod v$, $c \neq b_v$.*

Proof. Again, the value of c determines two cases.

Case 1: Suppose $v \equiv 4 \pmod{12}$ and $c \equiv 0 \pmod{2v}$, say $v = 12m + 4$, $m \geqslant 1$, and $c = 2kv$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant m, \\ m & \text{if } k > m, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant m, \\ k - m & \text{if } k > m. \end{cases}$$

Consider the set:

$$\{T(0, 3m + 1, 9m + 3)\}$$

$$\cup \{C(0, 1 + 2i, 5m + 2 + i), C(0, 12m + 3 - 2i, 7m + 2 - i) \mid i = 0, 1, \ldots, k_1 - 1$$
$$\text{(omit these triples if } k_1 = 0)\}$$

$$\cup \{T(0, 1 + 2i, 5m + 2 + i), T(0, 12m + 3 - 2i, 7m + 2 - i) \mid i = k_1, k_1 + 1, \ldots,$$
$$m - 1 \text{ (omit these triples if } k_1 = m)\}$$

$\cup\{C(0, 2+2i, 3m+2+i), C(0, 12m+2-2i, 9m+2-i) \mid i = 0, 1, \ldots, k_2 - 1$ (omit these triples if $k_2 = 0$)$\}$

$\cup\{T(0, 2+2i, 3m+2+i), T(0, 12m+2-2i, 9m+2-i) \mid i = k_2, k_2+1, \ldots, m-1$ (omit these triples if $k_2 = m$)$\}$.

Case 2: Suppose $v \equiv 4 \pmod{12}$ and $c \equiv v \pmod{2v}$, say $v = 12m + 4$ and $c = (2k+1)v$. Then since $c \in B_v$, we have $0 \leqslant k \leqslant 2m$. If $k = 2m$, and therefore $c = b_v$, then the c-$HTS(v)$ is actually an $MTS(v)$, and it is known that a cyclic $MTS(v)$ does not exist. So we consider the case $0 \leqslant k \leqslant 2m - 1$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant m-1, \\ m-1 & \text{if } k > m-1, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant m-1, \\ k-m+1 & \text{if } k > m-1. \end{cases}$$

Consider the set:

$\{T(0, 5m+2, 11m+4), T(0, 2m+1, 10m+3), C(0, 3m+1, 11m+4)\}$

$\cup\{C(0, 1+i, 2m+3+2i), C(0, 6m+1-i, 6m-2i) \mid i = 0, 1, \ldots, k_1 - 1$ (omit these triples if $k_1 = 0$ or if $m = 1$)$\}$

$\cup\{T(0, 2m+2+i, 12m+3-i), T(0, 6m+1-i, 1+i) \mid i = k_1, k_1+1, \ldots, m-2$ (omit these triples if $k_1 = m-1$ or if $m = 1$)$\}$

$\cup\{C(0, m+1+i, 4m+3+2i), C(0, 5m+1-i, 4m-2i) \mid i = 0, 1, \ldots, k_2 - 1$ (omit these triples if $k_2 = 0$)$\}$

$\cup\{T(0, 3m+2+i, 11m+3-i), T(0, 5m+1-i, m+1+i) \mid i = k_2, k_2+1, \ldots, m-1$ (omit these triples if $k_2 = m$)$\}$.

In both cases, the given set is a set of base triples for a cyclic c-$HTS(v)$ under the permutation $\pi = (0, 1, \ldots, v-1)$. □

We now consider the case $v \equiv 0 \pmod 3$. Here, $b_v = v(v-1)/3 \equiv 2v/3 \pmod v$. Since $b_v = t + c$ and $t \equiv 0 \pmod v$ by Lemma 2.1, we have:

Lemma 2.5. *If a cyclic c-$HTS(v)$ exists where $v \equiv 0 \pmod 3$, then $c \equiv 2v/3 \pmod v$.*

Lemma 2.6. *If $v \equiv 3 \pmod 6$, $v \neq 9$, then there exists a cyclic c-$HTS(v)$ for all $c \in B_v$ such that $c \equiv 2v/3 \pmod v$ except for $c = v(v-4)/3$.*

Proof. First we note that by the argument of case 2 of Lemma 2.3, a cyclic $v(v-4)/3$-$HTS(v)$ cannot exist.

Case 1: Suppose $v \equiv 3 \pmod 6$, $v \neq 9$, and $c \equiv 2v/3 \pmod{2v}$. Then there exists a cyclic Steiner triple system of order v [11] and the result follows as in case 1 of Lemma 2.3.

Case 2: Suppose $v \equiv 3 \pmod{36}$ and $c \equiv 5v/3 \pmod{2v}$, say $v = 36m + 3$, $m \geqslant 1$, and $c = 2kv + 5v/3$. In this case we have $0 \leqslant k \leqslant 6m - 2$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant 2m - 1, \\ 2m - 1 & \text{if } k > 2m - 1, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant 2m - 1, \\ k - 2m + 1 & \text{if } 2m - 1 < k \leqslant 4m - 2, \\ 2m - 1 & \text{if } k > 4m - 2, \end{cases}$$

$$k_3 = \begin{cases} 0 & \text{if } k \leqslant 4m - 2, \\ k - 4m + 2 & \text{if } 4m - 2 < k \leqslant 5m - 2, \\ m & \text{if } k > 5m - 2, \end{cases}$$

$$k_4 = \begin{cases} 0 & \text{if } k \leqslant 5m - 2, \\ k - 5m + 2 & \text{if } 5m - 2 < k \leqslant 6m - 3, \\ m - 1 & \text{if } k = 6m - 2. \end{cases}$$

Consider the set

$$\{C(0, 8m, 22m + 1), C(0, 12m + 1, 24m + 2), C(0, 24m + 2, 12m + 1)\}$$

$$\cup \{T(0, 6m, 28m + 2), T(0, 28m + 3, 28m + 1), T(0, 8m + 1, 16m + 3)\}$$

$$\cup \{C(0, 5 + 3i, 8m + 4 + 2i), C(0, 36m - 2 - 3i, 28m - 1 - 2i) \mid i = 0, 1, \ldots, k_1 - 1 \text{ (omit these triples if } k_1 = 0)\}$$

$$\cup \{T(0, 5 + 3i, 8m + 4 + 2i), T(0, 36m - 2 - 3i, 28m - 1 - 2i) \mid i = k_1, k_1 + 1, \ldots, 2m - 2 \text{ (omit these triples if } k_1 = 2m - 1)\}$$

$$\cup \{C(0, 3 + 3i, 12m + 2 + i), C(0, 36m - 3i, 24m + 1 - i) \mid i = 0, 1, \ldots, k_2 - 1 \text{ (omit these triples if } k_2 = 0)\}$$

$$\cup \{T(0, 3 + 3i, 12m + 2 + i), T(0, 36m - 3i, 24m + 1 - i) \mid i = k_2, k_2 + 1, \ldots, 2m - 2 \text{ (omit these triples if } k_2 = 2m - 1)\}$$

$$\cup \{C(0, 1 + 3i, 16m + 2 + 2i), C(0, 36m + 2 - 3i, 20m + 1 - 2i) \mid i = 0, 1, \ldots, k_3 - 1 \text{ (omit these triples if } k_3 = 0)\}$$

$$\cup \{T(0, 1 + 3i, 16m + 2 + 2i), T(0, 36m + 2 - 3i, 20m + 1 - 2i) \mid i = k_3, k_3 + 1, \ldots, m - 1 \text{ (omit these triples if } k_3 = m)\}$$

$$\cup \{C(0, 6m - 5 - 3i, 20m - 2 - 2i), C(0, 30m + 8 + 3i, 16m + 5 + 2i) \mid i = 0, 1, \ldots, k_4 - 1 \text{ (omit these triples if } k_4 = 0 \text{ or if } m = 1)\}$$

$$\cup \{T(0, 6m - 5 - 3i, 20m - 2 - 2i), T(0, 30m + 8 + 3i, 16m + 5 + 2i) \mid i = k_4, k_4 + 1, \ldots, m - 2 \text{ (omit these triples if } k_4 = m - 1 \text{ or if } m = 1)\}$$

$\cup\{C(0,2,30m+5), C(0,20m,6m-2)$ (omit these triples if $k<6m-2)\}$

$\cup\{T(0,2,30m+5), T(0,20m,6m-2)$ (omit these triples if $k=6m-2)\}$.

Case 3: Suppose $v \equiv 9 \pmod{36}$ and $c \equiv 5v/3 \pmod{2v}$, say $v = 36m+9$, $m \geqslant 1$, and $c = 2kv + 5v/3$. In this case we have $0 \leqslant k \leqslant 6m-1$. If $m=1$, consider the set:

$\{C(0,21,9), C(0,15,30), C(0,30,15), T(0,24,18), T(0,44,11), T(0,34,16)\}$

$\cup\{C(0,2,19), C(0,43,26)$ (omit these triples if $k=0)\}$

$\cup\{T(0,2,19), T(0,43,26)$ (omit these triples if $k \geqslant 1)\}$

$\cup\{C(0,3,23), C(0,42,22)$ (omit these triples if $k<2)\}$

$\cup\{T(0,3,23), T(0,42,22)$ (omit these triples if $k \geqslant 2)\}$

$\cup\{C(0,4,14), C(0,41,31)$ (omit these triples if $k<3)\}$

$\cup\{T(0,4,14), T(0,41,31)$ (omit these triples if $k \geqslant 3)\}$

$\cup\{C(0,5,13), C(0,40,32)$ (omit these triples if $k<4)\}$

$\cup\{T(0,5,13), T(0,40,32)$ (omit these triples if $k \geqslant 4)\}$

$\cup\{C(0,1,7), C(0,7,16)$ (omit these triples if $k<5)\}$

$\cup\{T(0,1,7), T(0,9,38)$ (omit these triples if $k=5)\}$.

Now suppose $m>1$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant 2m, \\ 2m & \text{if } k > 2m, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant 2m, \\ k-2m & \text{if } 2m < k \leqslant 4m-3, \\ 2m-3 & \text{if } k > 4m-3, \end{cases}$$

$$k_3 = \begin{cases} 0 & \text{if } k \leqslant 4m-3, \\ k-4m+3 & \text{if } 4m-3 < k \leqslant 5m-4, \\ m-1 & \text{if } k > 5m-4, \end{cases}$$

$$k_4 = \begin{cases} 0 & \text{if } k \leqslant 5m-4, \\ k-5m+4 & \text{if } 5m-4 < k \leqslant 6m-7, \\ m-3 & \text{if } k > 6m-7. \end{cases}$$

Consider the set

$\{C(0,20m,14m+4), C(0,12m+3,24m+6), C(0,24m+6,12m+3)\}$

$\cup\{T(0,1,6m-3), T(0,16m+9,8m+5), T(0,28m+4,22m+7)\}$

$\cup\{C(0, 4+3i, 8m+6+2i), C(0, 36m+5-3i, 28m+3-2i) \mid$
$i = 0, 1, \ldots, k_1 - 1$ (omit these triples if $k_1 = 0$)$\}$

$\cup\{T(0, 4+3i, 8m+6+2i), T(0, 36m+5-3i, 28m+3-2i) \mid$
$i = k_1, k_1+1, \ldots, 2m-1$ (omit these triples if $k_1 = 2m$)$\}$

$\cup\{C(0, 6+3i, 12m+5+i), C(0, 36m+3-3i, 24m+4-i) \mid$
$i = 0, 1, \ldots, k_2 - 1$ (omit these triples if $k_2 = 0$)$\}$

$\cup\{T(0, 6+3i, 12m+5+i), T(0, 36m+3-3i, 24m+4-i) \mid$
$i = k_2, k_2+1, \ldots, 2m-4$ (omit these triples if $k_2 = 2m-3$)$\}$

$\cup\{C(0, 8+3i, 16m+8+2i), C(0, 36m+1-3i, 20m+1-2i) \mid$
$i = 0, 1, \ldots, k_3 - 1$ (omit these triples if $k_3 = 0$ or if $m < 3$)$\}$

$\cup\{T(0, 8+3i, 16m+8+2i), T(0, 36m+1-3i, 20m+1-2i) \mid$
$i = k_3, k_3+1, \ldots, m-2$ (omit these triples if $k_3 = m-1$ or if $m < 3$)$\}$

$\cup\{C(0, 6m-7-3i, 20m-2-2i), C(0, 30m+16+3i, 16m+11+2i) \mid$
$i = 0, 1, \ldots, k_4 - 1$ (omit these triples if $k_4 = 0$ or if $m < 4$)$\}$

$\cup\{T(0, 6m-7-3i, 20m-2-2i), T(0, 30m+16+3i, 16m+11+2i) \mid$
$i = k_4, k_4+1, \ldots, m-4$ (omit these triples if $k_4 = m-3$ or if $m < 4$)$\}$

$\cup\{C(0, 6m-1, 12m+1), C(0, 30m+10, 24m+8)$
(omit these triples if $k \leqslant 6m-7$)$\}$

$\cup\{T(0, 6m-1, 12m+1), T(0, 30m+10, 24m+8)$
(omit these triples if $k > 6m-7$)$\}$

$\cup\{C(0, 6m, 20m+3), C(0, 30m+9, 16m+6)$
(omit these triples if $k \leqslant 6m-6$)$\}$

$\cup\{T(0, 6m, 20m+3), T(0, 30m+9, 16m+6)$
(omit these triples if $k > 6m-6$)$\}$

$\cup\{C(0, 3, 16m+4), C(0, 36m+6, 20m+5)$
(omit these triples if $k \leqslant 6m-5$)$\}$

$\cup\{T(0, 3, 16m+4), T(0, 36m+6, 20m+5)$
(omit these triples if $k > 6m-5$)$\}$

$\cup\{C(0, 2, 16m+5), C(0, 36m+7, 20m+4)$
(omit these triples if $k \leqslant 6m-4$)$\}$

$\cup\{T(0, 2, 16m+5), T(0, 36m+7, 20m+4)$
(omit these triples if $k > 6m-4$)$\}$

$\cup\{C(0, 5, 16m+7), C(0, 36m+4, 20m+2)$
(omit these triples if $k \leqslant 6m-3$)$\}$

$\cup\{T(0,5,16m+7), T(0,36m+4,20m+2)$
(omit these triples if $k>6m-3)\}$

$\cup\{C(0,8m+4,8m+3), C(0,14m+2,28m+6)$
(omit these triples if $k\leqslant 6m-2)\}$

$\cup\{T(0,8m+4,8m+3), T(0,14m+2,28m+6)$
(omit these triples if $k=6m-1)\}$.

Case 4: Suppose $v\equiv 15 \pmod{36}$ and $c\equiv 5v/3 \pmod{2v}$, say $v=36m+15, m\geqslant 0$, and $c=2kv+5v/3$. In this case we have $0\leqslant k\leqslant 6m$. If $m=0$, consider the set

$$\{C(0,1,7), C(0,5,10), C(0,10,5), T(0,2,11), T(0,3,7), T(0,14,12)\}.$$

Now suppose $m>0$. Define

$$k_1=\begin{cases} k & \text{if } k\leqslant 2m-1,\\ 2m-1 & \text{if } k>2m-1,\end{cases}$$

$$k_2=\begin{cases} 0 & \text{if } k\leqslant 2m-1,\\ k-2m+1 & \text{if } 2m-1<k\leqslant 4m-1,\\ 2m & \text{if } k>4m-1,\end{cases}$$

$$k_3=\begin{cases} 0 & \text{if } k\leqslant 4m-1,\\ k-4m+1 & \text{if } 4m-1<k\leqslant 5m-1,\\ m & \text{if } k>5m-1,\end{cases}$$

$$k_4=\begin{cases} 0 & \text{if } k\leqslant 5m-1,\\ k-5m+1 & \text{if } 5m-1<k\leqslant 6m-1,\\ m & \text{if } k=6m.\end{cases}$$

Consider the set:

$\{C(0,6m+2,1), C(0,12m+5,24m+10), C(0,24m+10,12m+5)\}$

$\cup\{T(0,1,28m+12), T(0,6m+1,30m+12), T(0,6m+3,14m+6)\}$

$\cup\{C(0,4+3i,8m+6+2i), C(0,36m+11-3i,28m+9-2i)|$
$i=0,1,\ldots,k_1-1$ (omit these triples if $k_1=0)\}$

$\cup\{T(0,4+3i,8m+6+2i), T(0,36m+11-3i,28m+9-2i)|$
$i=k_1,k_1+1,\ldots,2m-2$ (omit these triples if $k_1=2m-1)\}$

$\cup\{C(0,3+3i,12m+6+i), C(0,36m+12-3i,24m+9-i)|$
$i=0,1,\ldots,k_2-1$ (omit these triples if $k_2=0)\}$

$\cup\{T(0,3+3i,12m+6+i), T(0,36m+12-3i,24m+9-i)|$
$i=k_2,k_2+1,\ldots,2m-1$ (omit these triples if $k_2=2m)\}$

$$\cup\{C(0, 2+3i, 16m+8+2i),\ C(0, 36m+13-3i, 20m+7-2i) \mid i = 0, 1, \ldots, k_3 - 1 \text{ (omit these triples if } k_3 = 0)\}$$

$$\cup\{T(0, 2+3i, 16m+8+2i),\ T(0, 36m+13-3i, 20m+7-2i) \mid i = k_3, k_3+1, \ldots, m-1 \text{ (omit these triples if } k_3 = m)\}$$

$$\cup\{C(0, 6m-1-3i, 20m+6-2i),\ C(0, 30m+16+3i, 16m+9+2i) \mid i = 0, 1, \ldots, k_4 - 1 \text{ (omit these triples if } k_4 = 0)\}$$

$$\cup\{T(0, 6m-1-3i, 20m+6-2i),\ T(0, 30m+16+3i, 16m+9+2i) \mid i = k_4, k_4+1, \ldots, m-1 \text{ (omit these triples if } k_4 = m)\}$$

$$\cup\{C(0, 8m+4, 20m+8),\ C(0, 22m+9, 16m+7) \text{ (omit these triples if } k < 6m)\}$$

$$\cup\{T(0, 8m+4, 20m+8),\ T(0, 22m+9, 16m+7) \text{ (omit these triples if } k = 6m)\}.$$

Case 5: Suppose $v \equiv 21 \pmod{36}$ and $c \equiv 5v/3 \pmod{2v}$, say $v = 36m + 21$, $m \geqslant 0$, and $c = 2kv + 5v/3$. In this case we have $0 \leqslant k \leqslant 6m + 1$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant 2m, \\ 2m & \text{if } k > 2m, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant 2m, \\ k - 2m & \text{if } 2m < k \leqslant 4m, \\ 2m & \text{if } k > 4m, \end{cases}$$

$$k_3 = \begin{cases} 0 & \text{if } k \leqslant 4m, \\ k - 4m & \text{if } 4m < k \leqslant 5m+1, \\ m+1 & \text{if } k > 5m+1, \end{cases}$$

$$k_4 = \begin{cases} 0 & \text{if } k \leqslant 5m+1, \\ k - 5m - 1 & \text{if } 5m+1 < k \leqslant 6m, \\ m-1 & \text{if } k = 6m+1. \end{cases}$$

Consider the set

$$\{C(0, 14m+8, 6m+3),\ C(0, 12m+7, 24m+14),\ C(0, 24m+14, 12m+7)\}$$

$$\cup\{T(0, 28m+17, 28m+15),\ T(0, 6m+3, 2),\ T(0, 22m+12, 8m+4)\}$$

$$\cup\{C(0, 5+3i, 8m+8+2i),\ C(0, 36m+16-3i, 28m+13-2i) \mid i = 0, 1, \ldots, k_1 - 1 \text{ (omit these triples if } k_1 = 0 \text{ or if } m = 0)\}$$

$$\cup\{T(0, 5+3i, 8m+8+2i),\ T(0, 36m+16-3i, 28m+13-2i) \mid i = k_1, k_1+1, \ldots, 2m-1 \text{ (omit these triples if } k_1 = 2m \text{ or if } m = 0)\}$$

$$\cup\{C(0,3+3i,12m+8+i),\ C(0,36m+18-3i,24m+13-i)\mid i=0,1,\ldots,k_2-1 \text{ (omit these triples if } k_2=0 \text{ or if } m=0)\}$$

$$\cup\{T(0,3+3i,12m+8+i),\ T(0,36m+18-3i,24m+13-i)\mid i=k_2,k_2+1,\ldots,2m-1 \text{ (omit these triples if } k_2=2m \text{ or if } m=0)\}$$

$$\cup\{C(0,1+3i,16m+10+2i),\ C(0,36m+20-3i,20m+11-2i)\mid i=0,1,\ldots,k_3-1 \text{ (omit these triples if } k_3=0 \text{ or if } m=0)\}$$

$$\cup\{T(0,1+3i,16m+10+2i),\ T(0,36m+20-3i,20m+11-2i)\mid i=k_3,k_3+1,\ldots,m \text{ (omit these triples if } k_3=m+1 \text{ or if } m=0)\}$$

$$\cup\{C(0,6m-2-3i,20m+8-2i),\ C(0,30m+23+3i,16m+13+2i)\mid i=0,1,\ldots,k_4-1 \text{ (omit these triples if } k_4=0 \text{ or if } m\leqslant 1)\}$$

$$\cup\{T(0,6m-2-3i,20m+8-2i),\ T(0,30m+23+3i,16m+13+2i)\mid i=k_4,k_4+1,\ldots,m-2 \text{ (omit these triples if } k_4=m-1 \text{ or if } m\leqslant 1)\}$$

$$\cup\{C(0,8m+5,16m+11),\ C(0,6m+1,20m+10) \text{ (omit these triples if } k<6m+1)\}$$

$$\cup\{T(0,8m+5,16m+11),\ T(0,6m+1,20m+10) \text{ (omit these triples if } k=6m+1)\}.$$

Case 6: Suppose $v\equiv 27 \pmod{36}$ and $c\equiv 5v/3 \pmod{2v}$, say $v=36m+27$, $m\geqslant 0$, and $c=2kv+5v/3$. In this case we have $0\leqslant k\leqslant 6m+2$. If $m=0$ and $k\leqslant 1$, consider the set:

$$\{C(0,10,3),\ C(0,9,18),\ C(0,18,9)\}$$
$$\cup\{T(0,2,7),\ T(0,3,11),\ T(0,6,4),\ T(0,16,15),\ T(0,21,17)\}$$
$$\cup\{C(0,1,13),\ C(0,19,14) \text{ (omit these triples if } k=0)\}$$
$$\cup\{T(0,1,13),\ T(0,19,14) \text{ (omit these triples if } k=1)\}.$$

If $m=0$ and $k=2$, consider the set

$$\{C(0,10,3), C(0,9,18), C(0,18,9), C(0,1,13), C(0,3,11), C(0,6,2), C(0,13,5), T(0,2,7), T(0,4,21), T(0,11,26)\}.$$

Suppose that $m=1$. Define

$$k_1=\begin{cases} k & \text{if } k\leqslant 4,\\ 4 & \text{if } k>4.\end{cases}$$

Consider the set

$$\{C(0,30,2),\ C(0,21,42),\ C(0,42,21),\ T(0,33,9)\}$$
$$\cup\{C(0,3,27),\ C(0,34,9),\ T(0,2,29),\ T(0,60,25) \text{ (omit these triples if } k<8)\}$$
$$\cup\{T(0,38,29),\ T(0,2,36),\ T(0,3,27),\ T(0,28,25) \text{ (omit these triples if } k=8)\}$$

$\cup\{C(0, 1+3i, 16+2i), C(0, 62-3i, 47-2i) \mid i = 0, 1, \ldots, k_1 - 1$
(omit these triples if $k_1 = 0$)$\}$

$\cup\{T(0, 1+3i, 16+2i), T(0, 62-3i, 47-2i) \mid i = k_1, k_1+1, \ldots, 3$
(omit these triples if $k_1 = 4$)$\}$

$\cup\{C(0,5,31), C(0,58,32)$ (omit these triples if $k < 5$)$\}$

$\cup\{T(0,5,31), T(0,58,32)$ (omit these triples if $k \geqslant 5$)$\}$

$\cup\{C(0,6,23), C(0,57,40)$ (omit these triples if $k < 6$)$\}$

$\cup\{T(0,6,23), T(0,57,40)$ (omit these triples if $k \geqslant 6$)$\}$

$\cup\{C(0,8,19), C(0,55,44)$ (omit these triples if $k < 7$)$\}$

$\cup\{T(0,8,19), T(0,55,44)$ (omit these triples if $k \geqslant 7$)$\}$.

Now suppose $m > 1$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant 2m+1, \\ 2m+1 & \text{if } k > 2m+1, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant 2m+1, \\ k-2m-1 & \text{if } 2m+1 < k \leqslant 4m-1, \\ 2m-2 & \text{if } k > 4m-1, \end{cases}$$

$$k_3 = \begin{cases} 0 & \text{if } k \leqslant 4m-1, \\ k-4m+1 & \text{if } 4m-1 < k \leqslant 5m-2, \\ m-1 & \text{if } k > 5m-2, \end{cases}$$

$$k_4 = \begin{cases} 0 & \text{if } k \leqslant 5m-2, \\ k-5m+2 & \text{if } 5m-2 < k \leqslant 6m-4, \\ m-2 & \text{if } k > 6m-4. \end{cases}$$

Consider the set

$\{C(0, 20m+10, 14m+11), C(0, 12m+9, 24m+18), C(0, 24m+18, 12m+9)\}$

$\cup\{T(0, 1, 6m), T(0, 16m+17, 8m+9), T(0, 28m+18, 22m+18)\}$

$\cup\{C(0, 4+3i, 8m+10+2i), C(0, 36m+23-3i, 28m+17-2i) \mid$
$i = 0, 1, \ldots, k_1 - 1$ (omit these triples if $k_1 = 0$)$\}$

$\cup\{T(0, 4+3i, 8m+10+2i), T(0, 36m+23-3i, 28m+17-2i) \mid$
$i = k_1, k_1+1, \ldots, 2m$ (omit these triples if $k_1 = 2m+1$)$\}$

$\cup\{C(0, 6+3i, 12m+11+i), C(0, 36m+21-3i, 24m+16-i) \mid$
$i = 0, 1, \ldots, k_2 - 1$ (omit these triples if $k_2 = 0$)$\}$

$\cup\{T(0, 6+3i, 12m+11+i), T(0, 36m+21-3i, 24m+16-i) \mid$
$i = k_2, k_2+1, \ldots, 2m-3$ (omit these triples if $k_2 = 2m-2$)$\}$

$\cup\{C(0, 8+3i, 16m+16+2i), C(0, 36m+19-3i, 20m+11-2i) \mid$
$i = 0, 1, \ldots, k_3-1$ (omit these triples if $k_3 = 0$)$\}$

$\cup\{T(0, 8+3i, 16m+16+2i), T(0, 36m+19-3i, 20m+11-2i) \mid$
$i = k_3, k_3+1, \ldots, m-2$ (omit these triples if $k_3 = m-1$)$\}$

$\cup\{C(0, 6m-4-3i, 20m+8-2i), C(0, 30m+31+3i, 16m+19+2i) \mid$
$i = 0, 1, \ldots, k_4-1$ (omit these triples if $k_4 = 0$ or if $m = 2$)$\}$

$\cup\{T(0, 6m-4-3i, 20m+8-2i), T(0, 30m+31+3i, 16m+19+2i) \mid$
$i = k_4, k_4+1, \ldots, m-3$ (omit these triples if $k_4 = m-2$ or if $m = 2$)$\}$

$\cup\{C(0, 6m+2, 12m+7), C(0, 30m+25, 24m+20)$
(omit these triples if $k \leqslant 6m-4$)$\}$

$\cup\{T(0, 6m+2, 12m+7), T(0, 30m+25, 24m+20)$
(omit these triples if $k > 6m-4$)$\}$

$\cup\{C(0, 6m+3, 20m+13), C(0, 30m+24, 16m+14)$
(omit these triples if $k \leqslant 6m-3$)$\}$

$\cup\{T(0, 6m+3, 20m+13), T(0, 30m+24, 16m+14)$
(omit these triples if $k > 6m-3$)$\}$

$\cup\{C(0, 3, 16m+12), C(0, 36m+24, 20m+15)$
(omit these triples if $k \leqslant 6m-2$)$\}$

$\cup\{T(0, 3, 16m+12), T(0, 36m+24, 20m+15)$
(omit these triples if $k > 6m-2$)$\}$

$\cup\{C(0, 2, 16m+13), C(0, 36m+25, 20m+14)$
(omit these triples if $k \leqslant 6m-1$)$\}$

$\cup\{T(0, 2, 16m+13), T(0, 36m+25, 20m+14)$
(omit these triples if $k > 6m-1$)$\}$

$\cup\{C(0, 5, 16m+15), C(0, 36m+22, 20m+12)$
(omit these triples if $k \leqslant 6m$)$\}$

$\cup\{T(0, 5, 16m+15), T(0, 36m+22, 20m+12)$
(omit these triples if $k > 6m$)$\}$

$\cup\{C(0, 8m+8, 8m+7), C(0, 14m+9, 28m+20)$
(omit these triples if $k \leqslant 6m+1$)$\}$

$\cup\{T(0, 8m+8, 8m+7), T(0, 14m+9, 28m+20)$
(omit these triples if $k = 6m+2$)$\}$.

Case 7: Suppose $v \equiv 33 \pmod{36}$ and $c \equiv 5v/3 \pmod{2v}$, say $v = 36m + 33$, $m \geqslant 0$, and $c = 2kv + 5v/3$. In this case we have $0 \leqslant k \leqslant 6m + 3$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant 2m, \\ 2m & \text{if } k > 2m, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant 2m, \\ k - 2m & \text{if } 2m < k \leqslant 4m + 1, \\ 2m + 1 & \text{if } k > 4m + 1, \end{cases}$$

$$k_3 = \begin{cases} 0 & \text{if } k \leqslant 4m + 1, \\ k - 4m - 1 & \text{if } 4m + 1 < k \leqslant 5m + 2, \\ m + 1 & \text{if } k > 5m + 2, \end{cases}$$

$$k_4 = \begin{cases} 0 & \text{if } k \leqslant 5m + 2, \\ k - 5m - 2 & \text{if } 5m + 2 < k \leqslant 6m + 2, \\ m & \text{if } k = 6m + 3. \end{cases}$$

Consider the set:

$$\{C(0, 6m + 5, 1), C(0, 12m + 11, 24m + 22), C(0, 24m + 22, 12m + 11)\}$$

$$\cup \{T(0, 1, 28m + 26), T(0, 6m + 4, 30m + 27), T(0, 6m + 6, 14m + 13)\}$$

$$\cup \{C(0, 4 + 3i, 8m + 10 + 2i), C(0, 36m + 29 - 3i, 28m + 23 - 2i) \mid i = 0, 1, \ldots, k_1 - 1 \text{ (omit these triples if } k_1 = 0 \text{ or if } m = 0)\}$$

$$\cup \{T(0, 4 + 3i, 8m + 10 + 2i), T(0, 36m + 29 - 3i, 28m + 23 - 2i) \mid i = k_1, k_1 + 1, \ldots, 2m - 1 \text{ (omit these triples if } k_1 = 2m \text{ or if } m = 0)\}$$

$$\cup \{C(0, 3 + 3i, 12m + 12 + i), C(0, 36m + 30 - 3i, 24m + 21 - i) \mid i = 0, 1, \ldots, k_2 - 1 \text{ (omit these triples if } k_2 = 0)\}$$

$$\cup \{T(0, 3 + 3i, 12m + 12 + i), T(0, 36m + 30 - 3i, 24m + 21 - i) \mid i = k_2, k_2 + 1, \ldots, 2m \text{ (omit these triples if } k_2 = 2m + 1)\}$$

$$\cup \{C(0, 2 + 3i, 16m + 16 + 2i), C(0, 36m + 31 - 3i, 20m + 17 - 2i) \mid i = 0, 1, \ldots, k_3 - 1 \text{ (omit these triples if } k_3 = 0)\}$$

$$\cup \{T(0, 2 + 3i, 16m + 16 + 2i), T(0, 36m + 31 - 3i, 20m + 17 - 2i) \mid i = k_3, k_3 + 1, \ldots, m \text{ (omit these triples if } k_3 = m + 1)\}$$

$$\cup \{C(0, 6m + 2 - 3i, 20m + 16 - 2i), C(0, 30m + 31 + 3i, 16m + 17 + 2i) \mid i = 0, 1, \ldots, k_4 - 1 \text{ (omit these triples if } k_4 = 0 \text{ or if } m = 0)\}$$

$$\cup \{T(0, 6m + 2 - 3i, 20m + 16 - 2i), T(0, 30m + 31 + 3i, 16m + 17 + 2i) \mid i = k_4, k_4 + 1, \ldots, m - 1 \text{ (omit these triples if } k_4 = m \text{ or if } m = 0)\}$$

$$\cup\{C(0, 8m+8, 20m+18), C(0, 22m+20, 16m+15) \text{ (omit these triples if } k < 6m+3)\}$$

$$\cup\{T(0, 8m+8, 20m+18), T(0, 22m+20, 16m+15) \text{ (omit these triples if } k = 6m+3)\}.$$

In each case, the given set is a set of base triples for a cyclic c-$HTS(v)$ under the permutation $\pi = (0, 1, \ldots, v-1)$. □

Lemma 2.7. *If* $v \equiv 0 \pmod{12}$, *then there exists a cyclic* c-$HTS(v)$ *for all* $c \in B_v$ *such that* $c \equiv 2v/3 \pmod{v}$, *except for* $c = b_v$.

Proof.

Case 1: Suppose $v \equiv 0 \pmod{12}$ and $c \equiv 2v/3 \pmod{2v}$, say $v = 12m$, and $c = 2kv + 2v/3$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant m, \\ m & \text{if } k > m, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant m, \\ k-m & \text{if } k > m. \end{cases}$$

Consider the set:

$$\{C(0, 4m, 8m), C(0, 8m, 4m), T(0, 5m, 3m)\}$$

$$\cup\{C(0, 6m-i, 1+i), C(0, 2m+i, 12m-1-i) \mid i = 0, 1, \ldots, k_1-1 \text{ (omit these triples if } k_1 = 0)\}$$

$$\cup\{T(0, 6m-i, 1+i), T(0, 2m+i, 12m-1-i) \mid i = k_1, k_1+1, \ldots, m-1 \text{ (omit these triples if } k_1 = m)\}$$

$$\cup\{C(0, 5m-1-i, m+1+i), C(0, 3m+1+i, 11m-1-i) \mid i = 0, 1, \ldots, k_2-1 \text{ (omit these triples if } k_2 = 0 \text{ or if } m = 1)\}$$

$$\cup\{T(0, 5m-1-i, m+1+i), T(0, 3m+1+i, 11m-1-i) \mid i = k_2, k_2+1, \ldots, m-2 \text{ (omit these triples if } k_2 = m-1 \text{ or if } m = 1)\}.$$

Case 2: Suppose $v \equiv 0 \pmod{12}$ and $c \equiv 5v/3 \pmod{2v}$, say $v = 12m$, and $c = 2kv + 5v/3$. In the case of $c = b_v$, then a cyclic c-$HTS(v)$ would also be a cyclic $MTS(v)$, which does not exist. So we consider only $0 \leqslant k \leqslant 2m-2$. If $m = 1$ then consider the set

$$\{C(0, 4, 8), C(0, 8, 4), C(0, 6, 1), T(0, 10, 1), T(0, 9, 2)\}.$$

Now suppose $m > 1$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant m-1, \\ m-1 & \text{if } k > m-1, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant m-1, \\ k-m+1 & \text{if } k > m-1. \end{cases}$$

Consider the set

$$\{C(0,4m,8m), C(0,8m,4m), C(0,5m+1,m), T(0,10m,m), T(0,10m-1,2m)\}$$
$$\cup\{C(0,2m+1+i,12m-1-i), C(0,6m+1+2i,1+i) \mid i=0,1,\ldots,k_1-1 \text{ (omit these triples if } k_1=0)\}$$
$$\cup\{T(0,2m+1+i,12m-1-i), T(0,6m+1+2i,1+i) \mid i=k_1,k_1+1,\ldots,m-2 \text{ (omit these triples if } k_1=m-1)\}$$
$$\cup\{C(0,3m+1+i,11m-1-i), C(0,8m+1+2i,m+1+i) \mid i=0,1,\ldots,k_2-1 \text{ (omit these triples if } k_2=0)\}$$
$$\cup\{T(0,3m+1+i,11m-1-i), T(0,8m+1+2i,m+1+i) \mid i=k_2,k_2+1,\ldots,m-2 \text{ (omit these triples if } k_2=m-1)\}.$$

In both cases, the given set is a set of base triples for a cyclic c-$HTS(v)$ under the permutation $\pi = (0,1,\ldots,v-1)$. □

The results of this section combine to give us:

Theorem 2.1. *A cyclic c-$HTS(v)$ exists if and only if*

(i) $v \equiv 1 \pmod 6$ *and* $t \in \{0, 2v, 3v, 4v, \ldots, v(v-1)/3\}$, *or*
(ii) $v \equiv 4 \pmod{12}$ *and* $t \in \{v, 2v, 3v, \ldots, v(v-1)/3\}$, *or*
(iii) $v \equiv 3 \pmod 6$, $v \neq 9$ *and* $t \in \{0, 2v, 3v, 4v, \ldots, v(v-3)/3\}$, *or*
(iv) $v \equiv 0 \pmod{12}$ *and* $t \in \{v, 2v, 3v, \ldots, v(v-3)/3\}$

where $t = b_v - c = v(v-1)/3 - c$.

3. Rotational hybrid triple systems

In this section, we construct rotational c-$HTS(v)$s based on the vertex set $\{\infty\} \cup Z_{v-1}$ and admitting the automorphism $\rho = (\infty)\,(0,1,\ldots,v-2)$. We have the following necessary conditions:

Lemma 3.1. *If a rotational c-$HTS(v)$ exists, then* $v \equiv 0 \pmod 3$ *and* $c \equiv 0 \pmod{v-1}$ *or* $v \equiv 1 \pmod 3$ *and* $c \equiv (v-1)/3 \pmod{v-1}$. *Also, if* $v \equiv 0 \pmod 6$ *or if* $v = 10$, *then* $c \neq v(v-1)/3$.

Proof. The orbit of any transitive triple under ρ is of length $v-1$. Therefore, $t = v(v-1)/3 - c \equiv 0 \pmod{v-1}$. The result follows from a counting argument, along with the fact that a rotational $MTS(v)$ exists if and only if $v \equiv 1, 3$ or $4 \pmod 6$, $v \neq 10$. □

We now show that these necessary conditions are sufficient.

Lemma 3.2. *If* $v \equiv 4 \pmod 6$, *then there exists a rotational c-HTS*(v) *for all* $c \in B_v$ *such that* $c \equiv (v-1)/3 \pmod{v-1}$, *except for the case* $v = 10$ *and* $c = 30$.

Proof. First, suppose $v = 10$. If $c = 3$, consider the set $\{T(0, \infty, 7), C(0, 3, 6), T(0, 1, 5), T(0, 2, 8)\}$. If $c = 12$, consider the set $\{C(0, \infty, 2), C(0, 3, 6), T(0, 1, 5), T(0, 2, 8)\}$. If $c = 21$, consider the set $\{C(0, \infty, 7), C(0, 6, 3), T(0, 1, 5), C(0, 3, 1)\}$. In each case, the given set is a set of base triples for a rotational c-$HTS(10)$ under the permutation $\rho = (\infty)\,(0, 1, \ldots, 8)$.

Now suppose $v \equiv 4 \pmod 6$, $v \neq 10$. Then by case 1 of Lemma 2.6, there exists a cyclic c-$HTS(v-1)$ for all $c \equiv 2(v-1)/3 \pmod{2(v-1)}$. Let β^* be a set of base triples for such a design. Then β^* contains two triples, say b_1 and b_2, such that the orbits of b_1 and b_2 are each of length $(v-1)/3$ (notice then that b_1 is in the orbit of the block $C(0, (v-1)/3, 2(v-1)/3)$ and b_2 is in the orbit of the block $C(2(v-1)/3, (v-1)/3, 0)$). Consider the sets

$$\beta_1 = \{C(0, \infty, (v-1)/3), C(0, (v-1)/3, 2(v-1)/3)\} \cup \beta^* \setminus \{b_1, b_2\}$$

$$\beta_2 = \{T(0, \infty, 2(v-1)/3), C(0, (v-1)/3, 2(v-1)/3)\} \cup \beta^* \setminus \{b_1, b_2\}.$$

Then β_1 is a set of base triples for a rotational c-$HTS(v)$ where $c \equiv 4(v-1)/3 \pmod{2(v-1)}$ and β_2 is a set of base triples for a rotational c-$HTS(v)$ where $c \equiv (v-1)/3 \pmod{2(v-1)}$. Therefore, there exists a rotational c-$HTS(v)$ for all $c \equiv (v-1)/3 \pmod{v-1}$. □

Lemma 3.3. *If* $v \equiv 1 \pmod 6$, *then there exists a rotational c-HTS*(v) *for all* $c \in B_v$ *such that* $c \equiv (v-1)/3 \pmod{v-1}$.

Proof. We consider two cases.

Case 1: First, if $c = v(v-1)/3$ then a rotational c-$HTS(v)$ is also a rotational $MTS(v)$ and such a system is known to exist [1]. Now suppose $v \equiv 1 \pmod{12}$ and $c \equiv (v-1)/3 \pmod{v-1}$, say $v = 12m + 1$ and $c = (v-1)/3 + k(v-1)$ where $k < (v-1)/3$. Define

$$k_1 = \begin{cases} \lfloor k/2 \rfloor & \text{if } \lfloor k/2 \rfloor \leqslant m-1, \\ m-1 & \text{if } \lfloor k/2 \rfloor > m-1, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } \lfloor k/2 \rfloor \leqslant m-1, \\ \lfloor k/2 \rfloor - m + 1 & \text{if } m-1 < \lfloor k/2 \rfloor \leqslant 2m-2, \\ m-1 & \text{if } \lfloor k/2 \rfloor > 2m-2. \end{cases}$$

Consider the set

$$\{T(0,\ \infty,\ 8m)\ \text{(omit this triple if } k \text{ is odd)}\}$$
$$\cup\{C(0,\ \infty,\ 4m)\ \text{(omit this triple if } k \text{ is even)}\}\cup\{T(0, m, 3m), C(0, 4m, 8m)\}$$
$$\cup\{C(0, 10m, 8m-1), C(0, 5m+1, 4m+1)\ \text{(omit these triples if } k < 4m-2)\}$$
$$\cup\{T(0, 10m, 8m-1), T(0, 5m+1, 4m+1)\ \text{(omit these triples if } k \geqslant 4m-2)\}$$
$$\cup\{C(0, 2m+1+i, 12m-1-i), C(0, 6m-i, 1+i) \mid i = 0, 1, \ldots, k_1-1\ \text{(omit these triples if } k_1 = 0 \text{ or if } m = 1)\}$$
$$\cup\{T(0, 2m+1+i, 12m-1-i), T(0, 6m-i, 1+i) \mid i = k_1, k_1+1, \ldots, m-2\ \text{(omit these triples if } k_1 = m-1 \text{ or if } m = 1)\}$$
$$\cup\{C(0, 3m+1+i, 11m-1-i), C(0, 5m-i, m+1+i) \mid i = 0, 1, \ldots, k_2-1\ \text{(omit these triples if } k_2 = 0 \text{ or if } m = 1)\}$$
$$\cup\{T(0, 3m+1+i, 11m-1-i), T(0, 5m-i, m+1+i) \mid i = k_2, k_2+1, \ldots, m-2\ \text{(omit these triples if } k_2 = m-1 \text{ or if } m = 1)\}.$$

Case 2: First, if $c = v(v-1)/3$ then a rotational c-$HTS(v)$ is also a rotational $MTS(v)$ and such a system is known to exist [1]. Now suppose $v \equiv 7 \pmod{12}$ and $c \equiv (v-1)/3 \pmod{v-1}$, say $v = 12m+7$ and $c = (v-1)/3 + k(v-1)$ where $k < (v-1)/3$. Define

$$k_1 = \begin{cases} \lfloor k/2 \rfloor & \text{if } \lfloor k/2 \rfloor \leqslant m, \\ m & \text{if } \lfloor k/2 \rfloor > m, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } \lfloor k/2 \rfloor \leqslant m, \\ \lfloor k/2 \rfloor - m & \text{if } m < \lfloor k/2 \rfloor \leqslant 2m-1, \\ m-1 & \text{if } \lfloor k/2 \rfloor > 2m-1. \end{cases}$$

Consider the set:

$$\{T(0,\ \infty,\ 2m+1)\ \text{(omit this triple if } k \text{ is odd)}\}$$
$$\cup\{C(0,\ \infty,\ 10m+5)\ \text{(omit this triple if } k \text{ is even)}\}$$
$$\cup\{T(0, 11m+5, 4m+2), C(0, 8m+4, 4m+2)\}$$
$$\cup\{C(0, m+1, 4m+3), C(0, 10m+4, 8m+3)\ \text{(omit these triples if } k < 4m \text{ or if } m = 0)\}$$
$$\cup\{T(0, m+1, 4m+3), T(0, 10m+4, 8m+3)\ \text{(omit these triples if } k \geqslant 4m \text{ or if } m = 0)\}$$
$$\cup\{C(0, 2m+2+i, 12m+5-i), C(0, 6m+3-i, 1+i) \mid i = 0, 1, \ldots, k_1-1\ \text{(omit these triples if } k_1 = 0 \text{ or if } m = 0)\}$$

$\cup\{T(0, 2m+2+i, 12m+5-i), T(0, 6m+3-i, 1+i) \mid$
$i = k_1, k_1+1, \ldots, m-1$ (omit these triples if $k_1 = m$ or if $m = 0$)$\}$

$\cup\{C(0, 3m+3+i, 11m+4-i), C(0, 5m+2-i, m+2+i) \mid$
$i = 0, 1, \ldots, k_2-1$ (omit these triples if $k_2 = 0$ or if $m \leqslant 1$)$\}$

$\cup\{T(0, 3m+3+i, 11m+4-i), T(0, 5m+2-i, m+2+i) \mid$
$i = k_2, k_2+1, \ldots, m-2$ (omit these triples if $k_2 = m-1$ or if $m \leqslant 1$)$\}$.

In both cases, the given set is a set of base triples for a rotational c-$HTS(v)$ under the permutation $\rho = (\infty)\,(0, 1, \ldots, v-2)$. □

Lemma 3.4. *If $v \equiv 0 \pmod 3$, then there exists a rotational c-$HTS(v)$ for all $c \in B_v$ such that $c \equiv 0 \pmod{v-1}$, except the case of $v \equiv 0 \pmod 6$ and $c = v(v-1)/3$.*

Proof. We consider two cases.

Case 1: Suppose $v \equiv 0 \pmod 6$ and $c \equiv 0 \pmod{v-1}$, say $v = 6m$ and $c = k(v-1)$. Consider the set:

$\{T(0, \infty, 2m)$ (omit this triple if k is odd)$\}$

$\cup\{C(0, \infty, 4m-1)$ (omit this triple if k is even)$\} \cup \{T(0, 3m, 5m-1)\}$

$\cup\{C(0, m+i, 6m-2-i), C(0, 3m-1-i, 1+i) \mid$
$i = 0, 1, \ldots, \lfloor k/2 \rfloor - 1$ (omit these triples if $k = 0$ or 1, or if $m = 1$)$\}$

$\cup\{T(0, m+i, 6m-2-i), T(0, 3m-1-i, 1+i) \mid$
$i = \lfloor k/2 \rfloor, \lfloor k/2 \rfloor + 1, \ldots, m-2$ (omit these triples if $m = 1$)$\}$.

Case 2: Suppose $v \equiv 3 \pmod 6$ and $c \equiv 0 \pmod{v-1}$, say $v = 6m-3$, $m \geqslant 1$, and $c = k(v-1)$. Consider the set

$\{T(0, \infty, 3m-2)$ (omit this triple if k is odd)$\}$

$\cup\{C(0, \infty, 3m-2)$ (omit this triple if k is even)$\}$

$\cup\{C(0, m+i, 6m-5-i), C(0, 3m+2i, 1+i) \mid$
$i = 0, 1, \ldots, \lfloor k/2 \rfloor - 1$ (omit these triples if $k = 0$ or 1, or if $m = 1$)$\}$

$\cup\{T(0, m+i, 6m-5-i), T(0, 3m+2i, 1+i) \mid i = \lfloor k/2 \rfloor \lfloor k/2 \rfloor$
$+1, \ldots, m-2$ (omit these triples if $k = 2m-1$ or $m = 1$)$\}$.

In both cases, the given set is a set of base triples for a rotational c-$HTS(v)$ under the permutation $\rho = (\infty)\,(0, 1, \ldots, v-2)$. □

The results of this section combine to give us:

Theorem 3.1. *A rotational c-$HTS(v)$ exists if and only if*
(i) *$v = 10$ and $t \in \{9, 18, 27\}$, or*
(ii) *$v \equiv 1 \pmod 3$, $v \neq 10$, and $t \in \{0, (v-1), 2(v-1), 3(v-1), \ldots, (v-1)^2/3$, or*

(iii) $v \equiv 0 \pmod 6$ *and* $t \in \{(v-1), 2(v-1), 3(v-1), \ldots, v(v-1)/3\}$, *or*
(iv) $v \equiv 3 \pmod 6$ *and* $t \in \{0, (v-1), 2(v-1), 3(v-1), \ldots, v(v-1)/3\}$
where $t = b_v - c = v(v-1)/3 - c$.

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