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Cyclic and rotational hybrid triple systems

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Abstract

A hybrid triple system of order v, denoted HTS(v), is said to be *cyclic* if it admits an automorphism consisting of a single cycle of length v. A HTS(v) admitting an automorphism consisting of a fixed point and a cycle of length $v-1$ is said to be *rotational*. Necessary and sufficient conditions are given for the existence of a cyclic HTS(v) and a rotational HTS(v).

1. Introduction

A *cyclic triple*, denoted $C(a,b,c)$, is the digraph on the vertex set $\{a,b,c\}$ with the arc set $\{(a,b),(b,c),(c,a)\}$. A *transitive triple*, denoted $T(a,b,c)$, is the digraph on the vertex set $\{a,b,c\}$ with the arc set $\{(a,b),(b,c),(a,c)\}$. Let D_v denote the complete symmetric digraph on v vertices and let $b_v = v(v-1)/3$. A *c-hybrid triple system* of order v, denoted c-HTS(v), is an arc-disjoint partition of D_v into c cyclic triples and $t = b_v - c$ transitive triples.

A *directed triple system* of order v, denoted DTS(v), is a 0-HTS(v). A DTS(v) exists if and only if $v \equiv 0$ or $1 \pmod 3$ [7]. A *Mendelsohn triple system* of order v, denoted MTS(v), is a b_v-HTS(v). An MTS(v) exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$ [9]. In general, a c-HTS(v) exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$, and $c \in B_v = \{0,1,2,\ldots,b_v-2,b_v\}$ [5,6]. A related design is an *oriented triple system* (also called an *ordered triple system*) of order v, denoted OTS(v), which is a c-HTS(v) where c is any integer such that $0 \leqslant c \leqslant b_v$ (notice that c cannot equal $b_v - 1$). Therefore, an OTS(v) exists if and only if $v \equiv 0$ or $1 \pmod 3$ [8].

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An *automorphism* of a c-HTS(v) based on D_v is a permutation π of the vertex set of D_v which fixes the collection of triples of the c-HTS(v). The *orbit* of a triple under an automorphism π is the collection of images of the triple under the powers of π. A set of triples β is a *set of base triples* for a c-HTS(v) under the automorphism π if the orbits of the triples of β produce a set of triples for a c-HTS(v) and exactly one triple of β occurs in each orbit.

A c-HTS(v) admitting an automorphism consisting of a single cycle is said to be *cyclic*. A cyclic DTS(v) exists if and only if $v \equiv 1, 4$, or $7 \pmod{12}$ [4] and a cyclic MTS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$ [3]. A cyclic OTS(v) exists if and only if $v \equiv 0, 1, 3, 4, 7$, or $9 \pmod{12}$, $v \neq 9$ [10].

A c-HTS(v) admitting an automorphism consisting of a fixed point and a cycle of length $v-1$ is said to be *rotational*. A rotational DTS(v) exists if and only if $v \equiv 0 \pmod 3$ [2]. A rotational MTS(v) exists if and only if $v \equiv 1, 3$, or $4 \pmod 6$, $v \neq 10$ [1]. A rotational OTS(v) exists if and only if $v \equiv 0$ or $1 \pmod 3$ [10].

The purpose of this paper is to present necessary and sufficient conditions for the existence of cyclic and rotational c-HTS(v)'s.

2. Cyclic hybrid triple systems

In this section, we construct cyclic c-HTS(v)s based on the vertex set $\mathbf{Z}_v$ and admitting the automorphism $\pi = (0, 1, \ldots, v-1)$. The orbit of any transitive triple under π is of length v. We therefore immediately have

Lemma 2.1. *If a cyclic c-HTS(v) exists, then $t = b_v - c \equiv 0 \pmod v$.*

Since a cyclic c-HTS(v) is also an OTS(v), we have [10]

Lemma 2.2. *If a cyclic c-HTS(v) exists, then $v \equiv 0, 1, 3, 4, 7$, or $9 \pmod{12}, v \neq 9$.*

We give the remaining necessary conditions and establish sufficiency in the next five lemmas.

Lemma 2.3. *If $v \equiv 1 \pmod 6$ then there exists a cyclic c-HTS(v) for all $c \in B_v$ such that $c \equiv 0 \pmod v$, except for the case $c = v(v-4)/3$.*

Proof. We consider two cases based on the value of c.

Case 1: Suppose $v \equiv 1 \pmod 6$ and $c \equiv 0 \pmod{2v}$. Then there exists a cyclic Steiner triple system of order v [11]. Let B be a collection of base triples for such a system. Let $B = B_1 \cup B_2$ where $|B_1| = c/(2v)$. Define

$$H = \{C(a,b,c), C(c,b,a) \mid (a,b,c) \in B_1\} \cup \{T(a,b,c), T(c,b,a) \mid (a,b,c) \in B_2\}.$$

Then H is a set of base triples for a cyclic c-HTS(v) where $c \in B_v$ and $c \equiv 0 \pmod{2v}$.

Case 2: Suppose $v \equiv 1 \pmod 6$, say $v = 6m+1$, $m > 1$, and $c \equiv v \pmod{2v}$. If $c = v(v-4)/3$ then the existence of a cyclic c-HTS(v) is equivalent to partitioning the set $\{1,2,\ldots,6m\}$ into (difference) triples (x_i, y_i, z_i) for $i = 1,2,\ldots,2m-1$ and (x,y,z) such that $x_i + y_i + z_i \equiv 0 \pmod{6m+1}$ and $x + y \equiv z \pmod{6m+1}$. Now,

$$\sum_{i=1}^{6m} i = \sum_{i=1}^{2m-1} (x_i + y_i + z_i) + (x + y + z) = 3m(6m+1) \equiv 0 \pmod{6m+1},$$

and so $x + y + z \equiv 0 \pmod{6m+1}$. But $x + y + z \equiv 2c \pmod{6m+1}$. Therefore, $2z \equiv 0 \pmod{6m+1}$, which is impossible.

Suppose then that $c = (2k+1)v$ where $k = 0,1,\ldots,m-2$. Consider the set

$$\begin{aligned}&\{C(0,5m+1,1), T(0,5m,m), T(0,1,3m+1), T(0,3m+2,2m)\}\\ &\quad \cup \{C(0,m+2+i,6m-1-i), C(0,3m-1-i,2+i) \mid i = 0,1,\ldots,k-1\\ &\qquad \text{(omit these triples if } k = 0 \text{ or if } m = 2)\}\\ &\quad \cup \{T(0,m+2+i,6m-1-i), T(0,3m-1-i,2+i) \mid i = k,k+1,\ldots,m-3\\ &\qquad \text{(omit these triples if } k = m-2 \text{ or if } m = 2)\}.\end{aligned}$$

This is a set of base triples for a cyclic c-HTS(v) under the permutation $\pi = (0,1,\ldots,v-1)$. $\square$

Lemma 2.4. *If* $v \equiv 4 \pmod{12}$, *then there exists a cyclic* c-HTS(v) *for all* $c \in B_v$ *such that* $c \equiv 0 \pmod v$, $c \neq b_v$.

Proof. Again, the value of c determines two cases.

Case 1: Suppose $v \equiv 4 \pmod{12}$ and $c \equiv 0 \pmod{2v}$, say $v = 12m+4$, $m \geqslant 1$, and $c = 2kv$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant m,\\ m & \text{if } k > m,\end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant m,\\ k-m & \text{if } k > m.\end{cases}$$

Consider the set

$$\begin{aligned}&\{T(0,3m+1,9m+3)\}\\ &\quad \cup \{C(0,1+2i,5m+2+i), C(0,12m+3-2i,7m+2-i) \mid i = 0,1,\ldots,k_1-1\\ &\qquad \text{(omit these triples if } k_1 = 0)\}\\ &\quad \cup \{T(0,1+2i,5m+2+i), T(0,12m+3-2i,7m+2-i) \mid i = k_1,k_1+1,\ldots,m-1\\ &\qquad \text{(omit these triples if } k_1 = m)\}\\ &\quad \cup \{C(0,2+2i,3m+2+i), C(0,12m+2-2i,9m+2-i) \mid i = 0,1,\ldots,k_2-1\\ &\qquad \text{(omit these triples if } k_2 = 0)\}\end{aligned}$$

$$\cup \{T(0,2+2i,3m+2+i), T(0,12m+2-2i,9m+2-i) \mid i = k_2, k_2+1, \ldots, m-1$$
$$\text{(omit these triples if } k_2 = m)\}.$$

Case 2: Suppose $v \equiv 4 \pmod{12}$ and $c \equiv v \pmod{2v}$, say $v = 12m+4$ and $c = (2k+1)v$. Then since $c \in B_v$, we have $0 \leqslant k \leqslant 2m$. If $k = 2m$, and therefore $c = b_v$, then the c-HTS(v) is actually an MTS(v), and it is known that a cyclic MTS(v) does not exist. So we consider the case $0 \leqslant k \leqslant 2m-1$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant m-1, \\ m-1 & \text{if } k > m-1, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant m-1, \\ k-m+1 & \text{if } k > m-1. \end{cases}$$

Consider the set

$$\{T(0,5m+2,11m+4), T(0,2m+1,10m+3), C(0,3m+1,11m+4)\}$$
$$\cup \{C(0,1+i,2m+3+2i), C(0,6m+1-i,6m-2i) \mid i = 0,1,\ldots,k_1-1$$
$$\text{(omit these triples if } k_1 = 0 \text{ or if } m = 1)\}$$
$$\cup \{T(0,2m+2+i,12m+3-i), T(0,6m+1-i,1+i) \mid i = k_1, k_1+1, \ldots, m-2$$
$$\text{(omit these triples if } k_1 = m-1 \text{ or if } m = 1)\}$$
$$\cup \{C(0,m+1+i,4m+3+2i), C(0,5m+1-i,4m-2i) \mid i = 0,1,\ldots,k_2-1$$
$$\text{(omit these triples if } k_2 = 0)\}$$
$$\cup \{T(0,3m+2+i,11m+3-i), T(0,5m+1-i,m+1+i) \mid i = k_2, k_2+1, \ldots, m-1$$
$$\text{(omit these triples if } k_2 = m)\}.$$

In both cases, the given set is a set of base triples for a cyclic c-HTS(v) under the permutation $\pi = (0,1,\ldots,v-1)$. □

We now consider the case $v \equiv 0 \pmod 3$. Here, $b_v = v(v-1)/3 \equiv 2v/3 \pmod v$. Since $b_v = t + c$ and $t \equiv 0 \pmod v$ by Lemma 2.1, we have

Lemma 2.5. *If a cyclic c-HTS(v) exists where $v \equiv 0 \pmod 3$, then $c \equiv 2v/3 \pmod v$.*

Lemma 2.6. *If $v \equiv 3 \pmod 6$, $v \neq 9$, then there exists a cyclic c-HTS(v) for all $c \in B_v$ such that $c \equiv 2v/3 \pmod v$ except for $c = v(v-4)/3$.*

Proof. First we note that by the argument of case 2 of Lemma 2.3, a cyclic $v(v-4)/3$-HTS(v) cannot exist.

Case 1: Suppose $v \equiv 3 \pmod 6$, $v \neq 9$, and $c \equiv 2v/3 \pmod{2v}$. Then there exists a cyclic Steiner triple system of order v [11] and the result follows as in case 1 of Lemma 2.3.

Case 2: Suppose $v \equiv 3 \pmod{36}$ and $c \equiv 5v/3 \pmod{2v}$, say $v=36m+3$, $m \geqslant 1$, and $c = 2kv + 5v/3$. In this case we have $0 \leqslant k \leqslant 6m-2$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant 2m-1, \\ 2m-1 & \text{if } k > 2m-1, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant 2m-1, \\ k-2m+1 & \text{if } 2m-1 < k \leqslant 4m-2, \\ 2m-1 & \text{if } k > 4m-2, \end{cases}$$

$$k_3 = \begin{cases} 0 & \text{if } k \leqslant 4m-2, \\ k-4m+2 & \text{if } 4m-2 < k \leqslant 5m-2, \\ m & \text{if } k > 5m-2, \end{cases}$$

$$k_4 = \begin{cases} 0 & \text{if } k \leqslant 5m-2, \\ k-5m+2 & \text{if } 5m-2 < k \leqslant 6m-3, \\ m-1 & \text{if } k = 6m-2. \end{cases}$$

Consider the set

$$\begin{aligned}
&\{C(0,8m,22m+1), C(0,12m+1,24m+2), C(0,24m+2,12m+1)\} \\
&\quad \cup \{T(0,6m,28m+2), T(0,28m+3,28m+1), T(0,8m+1,16m+3)\} \\
&\quad \cup \{C(0,5+3i,8m+4+2i), C(0,36m-2-3i,28m-1-2i) \mid i=0,1,\ldots,k_1-1 \\
&\qquad \text{(omit these triples if } k_1=0)\} \\
&\quad \cup \{T(0,5+3i,8m+4+2i), T(0,36m-2-3i,28m-1-2i) \mid \\
&\qquad i=k_1,k_1+1,\ldots,2m-2 \text{ (omit these triples if } k_1=2m-1)\} \\
&\quad \cup \{C(0,3+3i,12m+2+i), C(0,36m-3i,24m+1-i) \mid i=0,1,\ldots,k_2-1 \\
&\qquad \text{(omit these triples if } k_2=0)\} \\
&\quad \cup \{T(0,3+3i,12m+2+i), T(0,36m-3i,24m+1-i) \mid i=k_2,k_2+1,\ldots,2m-2 \\
&\qquad \text{(omit these triples if } k_2=2m-1)\} \\
&\quad \cup \{C(0,1+3i,16m+2+2i), C(0,36m+2-3i,20m+1-2i) \mid i=0,1,\ldots,k_3-1 \\
&\qquad \text{(omit these triples if } k_3=0)\} \\
&\quad \cup \{T(0,1+3i,16m+2+2i), T(0,36m+2-3i,20m+1-2i) \mid \\
&\qquad i=k_3,k_3+1,\ldots,m-1 \text{ (omit these triples if } k_3=m)\} \\
&\quad \cup \{C(0,6m-5-3i,20m-2-2i), C(0,30m+8+3i,16m+5+2i) \mid \\
&\qquad i=0,1,\ldots,k_4-1 \text{ (omit these triples if } k_4=0 \text{ or if } m=1)\} \\
&\quad \cup \{T(0,6m-5-3i,20m-2-2i), T(0,30m+8+3i,16m+5+2i) \mid \\
&\qquad i=k_4,k_4+1,\ldots,m-2 \text{ (omit these triples if } k_4=m-1 \text{ or if } m=1)\} \\
&\quad \cup \{C(0,2,30m+5), C(0,20m,6m-2) \text{ (omit these triples if } k<6m-2)\} \\
&\quad \cup \{T(0,2,30m+5), T(0,20m,6m-2) \text{ (omit these triples if } k=6m-2)\}.
\end{aligned}$$

Case 3: Suppose $v \equiv 9 \pmod{36}$ and $c \equiv 5v/3 \pmod{2v}$, say $v = 36m + 9$, $m \geqslant 1$, and $c = 2kv + 5v/3$. In this case we have $0 \leqslant k \leqslant 6m - 1$. If $m = 1$, consider the set

$$\begin{aligned}
&\{C(0,21,9), C(0,15,30), C(0,30,15), T(0,24,18), T(0,44,11), T(0,34,16)\}\\
&\quad \cup \{C(0,2,19), C(0,43,26) \text{ (omit these triples if } k = 0)\}\\
&\quad \cup \{T(0,2,19), T(0,43,26) \text{ (omit these tiples if } k \geqslant 1)\}\\
&\quad \cup \{C(0,3,23), C(0,42,22) \text{ (omit these triples if } k < 2)\}\\
&\quad \cup \{T(0,3,23), T(0,42,22) \text{ (omit these triples if } k \geqslant 2)\}\\
&\quad \cup \{C(0,4,14), C(0,41,31) \text{ (omit these triples if } k < 3)\}\\
&\quad \cup \{T(0,4,14), T(0,41,31) \text{ (omit these triples if } k \geqslant 3)\}\\
&\quad \cup \{C(0,5,13), C(0,40,32) \text{ (omit these triples if } k < 4)\}\\
&\quad \cup \{T(0,5,13), T(0,40,32) \text{ (omit these triples if } k \geqslant 4)\}\\
&\quad \cup \{C(0,1,7), C(0,7,16) \text{ (omit these triples if } k < 5)\}\\
&\quad \cup \{T(0,1,7), T(0,9,38) \text{ (omit these triples if } k = 5)\}.
\end{aligned}$$

Now suppose $m > 1$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant 2m,\\ 2m & \text{if } k > 2m, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant 2m,\\ k - 2m & \text{if } 2m < k \leqslant 4m - 3,\\ 2m - 3 & \text{if } k > 4m - 3, \end{cases}$$

$$k_3 = \begin{cases} 0 & \text{if } k \leqslant 4m - 3,\\ k - 4m + 3 & \text{if } 4m - 3 < k \leqslant 5m - 4,\\ m - 1 & \text{if } k > 5m - 4, \end{cases}$$

$$k_4 = \begin{cases} 0 & \text{if } k \leqslant 5m - 4,\\ k - 5m + 4 & \text{if } 5m - 4 < k \leqslant 6m - 7,\\ m - 3 & \text{if } k > 6m - 7\ . \end{cases}$$

Consider the set

$$\begin{aligned}
&\{C(0,20m,14m+4), C(0,12m+3,24m+6), C(0,24m+6,12m+3)\}\\
&\quad \cup \{T(0,1,6m-3), T(0,16m+9,8m+5), T(0,28m+4,22m+7)\}\\
&\quad \cup \{C(0,4+3i,8m+6+2i), C(0,36m+5-3i,28m+3-2i) \mid\\
&\qquad i = 0,1,\ldots,k_1 - 1 \text{ (omit these triples if } k_1 = 0)\}
\end{aligned}$$

$\cup\{T(0,4+3i,8m+6+2i),T(0,36m+5-3i,28m+3-2i) \mid$
$i=k_1,k_1+1,\ldots,2m-1$(omit these triples if $k_1=2m$)$\}$
$\cup\{C(0,6+3i,12m+5+i),C(0,36m+3-3i,24m+4-i) \mid$
$i=0,1,\ldots,k_2-1$(omit these triples if $k_2=0$)$\}$
$\cup\{T(0,6+3i,12m+5+i),T(0,36m+3-3i,24m+4-i) \mid$
$i=k_2,k_2+1,\ldots,2m-4$ (omit these triples if $k_2=2m-3$)$\}$
$\cup\{C(0,8+3i,16m+8+2i),C(0,36m+1-3i,20m+1-2i) \mid$
$i=0,1,\ldots,k_3-1$(omit these triples if $k_3=0$ or if $m<3$)$\}$
$\cup\{T(0,8+3i,16m+8+2i),T(0,36m+1-3i,20m+1-2i) \mid$
$i=k_3,k_3+1,\ldots,m-2$ (omit these triples if $k_3=m-1$ or if $m<3$)$\}$
$\cup\{C(0,6m-7-3i,20m-2-2i),C(0,30m+16+3i,16m+11+2i) \mid$
$i=0,1,\ldots,k_4-1$ (omit these triples if $k_4=0$ or if $m<4$)$\}$
$\cup\{T(0,6m-7-3i,20m-2-2i),T(0,30m+16+3i,16m+11+2i) \mid$
$i=k_4,k_4+1,\ldots,m-4$ (omit these triples if $k_4=m-3$ or if $m<4$)$\}$
$\cup\{C(0,6m-1,12m+1),C(0,30m+10,24m+8)$ (omit these triples if $k\leqslant 6m-7$)$\}$
$\cup\{T(0,6m-1,12m+1),T(0,30m+10,24m+8)$(omit these triples if $k>6m-7$)$\}$
$\cup\{C(0,6m,20m+3),C(0,30m+9,16m+6)$ (omit these triples if $k\leqslant 6m-6$)$\}$
$\cup\{T(0,6m,20m+3),T(0,30m+9,16m+6)$ (omit these triples if $k>6m-6$)$\}$
$\cup\{C(0,3,16m+4),C(0,36m+6,20m+5)$ (omit these triples if $k\leqslant 6m-5$)$\}$
$\cup\{T(0,3,16m+4),T(0,36m+6,20m+5)$ (omit these triples if $k>6m-5$)$\}$
$\cup\{C(0,2,16m+5),C(0,36m+7,20m+4)$ (omit these triples if $k\leqslant 6m-4$)$\}$
$\cup\{T(0,2,16m+5),T(0,36m+7,20m+4)$ (omit these triples if $k>6m-4$)$\}$
$\cup\{C(0,5,16m+7),C(0,36m+4,20m+2)$ (omit these triples if $k\leqslant 6m-3$)$\}$
$\cup\{T(0,5,16m+7),T(0,36m+4,20m+2)$ (omit these triples if $k>6m-3$)$\}$
$\cup\{C(0,8m+4,8m+3),C(0,14m+2,28m+6)$ (omit these triples if $k\leqslant 6m-2$)$\}$
$\cup\{T(0,8m+4,8m+3),T(0,14m+2,28m+6)$ (omit these triples if $k=6m-1$)$\}$.

Case 4: Suppose $v \equiv 15 \pmod{36}$ and $c \equiv 5v/3 \pmod{2v}$, say $v = 36m+15$, $m \geqslant 0$, and $c = 2kv + 5v/3$. In this case we have $0 \leqslant k \leqslant 6m$. If $m = 0$, consider the set

$$\{C(0,1,7), C(0,5,10), C(0,10,5), T(0,2,11), T(0,3,7), T(0,14,12)\}.$$

Now suppose $m > 0$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant 2m-1, \\ 2m-1 & \text{if } k > 2m-1, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant 2m-1, \\ k-2m+1 & \text{if } 2m-1 < k \leqslant 4m-1, \\ 2m & \text{if } k > 4m-1, \end{cases}$$

$$k_3 = \begin{cases} 0 & \text{if } k \leqslant 4m-1, \\ k-4m+1 & \text{if } 4m-1 < k \leqslant 5m-1, \\ m & \text{if } k > 5m-1, \end{cases}$$

$$k_4 = \begin{cases} 0 & \text{if } k \leqslant 5m-1, \\ k-5m+1 & \text{if } 5m-1 < k \leqslant 6m-1, \\ m & \text{if } k = 6m\ . \end{cases}$$

Consider the set

$$\begin{aligned}
&\{C(0,6m+2,1), C(0,12m+5,24m+10), C(0,24m+10,12m+5)\} \\
&\quad \cup \{T(0,1,28m+12), T(0,6m+1,30m+12), T(0,6m+3,14m+6)\} \\
&\quad \cup \{C(0,4+3i,8m+6+2i), C(0,36m+11-3i,28m+9-2i) \mid \\
&\qquad i = 0,1,\ldots,k_1-1 \text{ (omit these triples if } k_1 = 0)\} \\
&\quad \cup \{T(0,4+3i,8m+6+2i), T(0,36m+11-3i,28m+9-2i) \mid \\
&\qquad i = k_1, k_1+1,\ldots,2m-2 \text{ (omit these triples if } k_1 = 2m-1)\} \\
&\quad \cup \{C(0,3+3i,12m+6+i), C(0,36m+12-3i,24m+9-i) \mid \\
&\qquad i = 0,1,\ldots,k_2-1 \text{ (omit these triples if } k_2 = 0)\} \\
&\quad \cup \{T(0,3+3i,12m+6+i), T(0,36m+12-3i,24m+9-i) \mid \\
&\qquad i = k_2, k_2+1,\ldots,2m-1 \text{ (omit these triples if } k_2 = 2m)\} \\
&\quad \cup \{C(0,2+3i,16m+8+2i), C(0,36m+13-3i,20m+7-2i) \mid \\
&\qquad i = 0,1,\ldots,k_3-1 \text{ (omit these triples if } k_3 = 0)\} \\
&\quad \cup \{T(0,2+3i,16m+8+2i), T(0,36m+13-3i,20m+7-2i) \mid \\
&\qquad i = k_3, k_3+1,\ldots,m-1 \text{ (omit these triples if } k_3 = m)\} \\
&\quad \cup \{C(0,6m-1-3i,20m+6-2i), C(0,30m+16+3i,16m+9+2i) \mid \\
&\qquad i = 0,1,\ldots,k_4-1 \text{ (omit these triples if } k_4 = 0)\}
\end{aligned}$$

$\cup \{T(0, 6m-1-3i, 20m+6-2i), T(0, 30m+16+3i, 16m+9+2i) \mid$
$i = k_4, k_4+1, \ldots, m-1$ (omit these triples if $k_4 = m$)$\}$
$\cup \{C(0, 8m+4, 20m+8), C(0, 22m+9, 16m+7)$ (omit these triples if $k < 6m$)$\}$
$\cup \{T(0, 8m+4, 20m+8), T(0, 22m+9, 16m+7)$ (omit these triples if $k = 6m$)$\}$.

Case 5: Suppose $v \equiv 21 \pmod{36}$ and $c \equiv 5v/3 \pmod{2v}$, say $v = 36m+21$, $m \geqslant 0$, and $c = 2kv + 5v/3$. In this case we have $0 \leqslant k \leqslant 6m+1$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant 2m, \\ 2m & \text{if } k > 2m, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant 2m, \\ k-2m & \text{if } 2m < k \leqslant 4m, \\ 2m & \text{if } k > 4m, \end{cases}$$

$$k_3 = \begin{cases} 0 & \text{if } k \leqslant 4m, \\ k-4m & \text{if } 4m < k \leqslant 5m+1, \\ m+1 & \text{if } k > 5m+1, \end{cases}$$

$$k_4 = \begin{cases} 0 & \text{if } k \leqslant 5m+1, \\ k-5m-1 & \text{if } 5m+1 < k \leqslant 6m, \\ m-1 & \text{if } k = 6m+1. \end{cases}$$

Consider the set

$\{C(0, 14m+8, 6m+3), C(0, 12m+7, 24m+14), C(0, 24m+14, 12m+7)\}$
$\cup \{T(0, 28m+17, 28m+15), T(0, 6m+3, 2), T(0, 22m+12, 8m+4)\}$
$\cup \{C(0, 5+3i, 8m+8+2i), C(0, 36m+16-3i, 28m+13-2i) \mid$
$i = 0, 1, \ldots, k_1-1$ (omit these triples if $k_1 = 0$ or if $m = 0$)$\}$
$\cup \{T(0, 5+3i, 8m+8+2i), T(0, 36m+16-3i, 28m+13-2i) \mid$
$i = k_1, k_1+1, \ldots, 2m-1$ (omit these triples if $k_1 = 2m$ or if $m = 0$)$\}$
$\cup \{C(0, 3+3i, 12m+8+i), C(0, 36m+18-3i, 24m+13-i) \mid$
$i = 0, 1, \ldots, k_2-1$ (omit these triples if $k_2 = 0$ or if $m = 0$)$\}$
$\cup \{T(0, 3+3i, 12m+8+i), T(0, 36m+18-3i, 24m+13-i) \mid$
$i = k_2, k_2+1, \ldots, 2m-1$ (omit these triples if $k_2 = 2m$ or if $m = 0$)$\}$
$\cup \{C(0, 1+3i, 16m+10+2i), C(0, 36m+20-3i, 20m+11-2i) \mid$
$i = 0, 1, \ldots, k_3-1$ (omit these triples if $k_3 = 0$ or if $m = 0$)$\}$

$\cup\{T(0,1+3i,16m+10+2i), T(0,36m+20-3i,20m+11-2i) \mid$
$i=k_3,k_3+1,\dots,m$ (omit these triples if $k_3=m+1$ or if $m=0$)$\}$
$\cup\{C(0,6m-2-3i,20m+8-2i), C(0,30m+23+3i,16m+13+2i) \mid$
$i=0,1,\dots,k_4-1$ (omit these triples if $k_4=0$ or if $m\leqslant 1$)$\}$
$\cup\{T(0,6m-2-3i,20m+8-2i), T(0,30m+23+3i,16m+13+2i) \mid$
$i=k_4,k_4+1,\dots,m-2$ (omit these triples if $k_4=m-1$ or if $m\leqslant 1$)$\}$
$\cup\{C(0,8m+5,16m+11), C(0,6m+1,20m+10)$ (omit these triples if $k<6m+1$)$\}$
$\cup\{T(0,8m+5,16m+11), T(0,6m+1,20m+10)$ (omit these triples if $k=6m+1$)$\}$.

Case 6: Suppose $v\equiv 27 \pmod{36}$ and $c\equiv 5v/3 \pmod{2v}$, say $v=36m+27$, $m\geqslant 0$, and $c=2kv+5v/3$. In this case we have $0\leqslant k\leqslant 6m+2$. If $m=0$ and $k\leqslant 1$, consider the set

$\{C(0,10,3), C(0,9,18), C(0,18,9)\}$
$\cup\{T(0,2,7), T(0,3,11), T(0,6,4), T(0,16,15), T(0,21,17)\}$
$\cup\{C(0,1,13), C(0,19,14)$ (omit these triples if $k=0$)$\}$
$\cup\{T(0,1,13), T(0,19,14)$ (omit these triples if $k=1$)$\}$.

If $m=0$ and $k=2$, consider the set

$\{C(0,10,3), C(0,9,18), C(0,18,9), C(0,1,13), C(0,3,11), C(0,6,2),$
$C(0,13,5), T(0,2,7), T(0,4,21), T(0,11,26)\}$.

Suppose that $m=1$. Define

$$k_1=\begin{cases} k & \text{if } k\leqslant 4,\\ 4 & \text{if } k>4.\end{cases}$$

Consider the set

$\{C(0,30,2), C(0,21,42), C(0,42,21), T(0,33,9)\}$
$\cup\{C(0,3,27), C(0,34,9), T(0,2,29), T(0,60,25)$ (omit these triples if $k<8$)$\}$
$\cup\{T(0,38,29), T(0,2,36), T(0,3,27), T(0,28,25)$ (omit these triples if $k=8$)$\}$
$\cup\{C(0,1+3i,16+2i), C(0,62-3i,47-2i) \mid i=0,1,\dots,k_1-1$
(omit these triples if $k_1=0$)$\}$

$$\cup \{T(0,1+3i,16+2i), T(0,62-3i,47-2i) \mid i = k_1, k_1+1, \ldots, 3$$
$$\text{(omit these triples if } k_1 = 4)\}$$
$$\cup \{C(0,5,31), C(0,58,32) \text{ (omit these triples if } k < 5)\}$$
$$\cup \{T(0,5,31), T(0,58,32) \text{ (omit these triples if } k \geqslant 5)\}$$
$$\cup \{C(0,6,23), C(0,57,40) \text{ (omit these triples if } k < 6)\}$$
$$\cup \{T(0,6,23), T(0,57,40) \text{ (omit these triples if } k \geqslant 6)\}$$
$$\cup \{C(0,8,19), C(0,55,44) \text{ (omit these triples if } k < 7)\}$$
$$\cup \{T(0,8,19), T(0,55,44) \text{ (omit these triples if } k \geqslant 7)\}.$$

Now suppose $m > 1$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant 2m+1, \\ 2m+1 & \text{if } k > 2m+1, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant 2m+1, \\ k-2m-1 & \text{if } 2m+1 < k \leqslant 4m-1, \\ 2m-2 & \text{if } k > 4m-1, \end{cases}$$

$$k_3 = \begin{cases} 0 & \text{if } k \leqslant 4m-1, \\ k-4m+1 & \text{if } 4m-1 < k \leqslant 5m-2, \\ m-1 & \text{if } k > 5m-2, \end{cases}$$

$$k_4 = \begin{cases} 0 & \text{if } k \leqslant 5m-2, \\ k-5m+2 & \text{if } 5m-2 < k \leqslant 6m-4, \\ m-2 & \text{if } k > 6m-4. \end{cases}$$

Consider the set

$$\{C(0,20m+10,14m+11), C(0,12m+9,24m+18), C(0,24m+18,12m+9)\}$$
$$\cup \{T(0,1,6m), T(0,16m+17,8m+9), T(0,28m+18,22m+18)\}$$
$$\cup \{C(0,4+3i,8m+10+2i), C(0,36m+23-3i,28m+17-2i) \mid$$
$$i = 0,1,\ldots,k_1-1 \text{ (omit these triples if } k_1 = 0)\}$$
$$\cup \{T(0,4+3i,8m+10+2i), T(0,36m+23-3i,28m+17-2i) \mid$$
$$i = k_1, k_1+1, \ldots, 2m \text{ (omit these triples if } k_1 = 2m+1)\}$$
$$\cup \{C(0,6+3i,12m+11+i), C(0,36m+21-3i,24m+16-i) \mid$$
$$i = 0,1,\ldots,k_2-1 \text{ (omit these triples if } k_2 = 0)\}$$
$$\cup \{T(0,6+3i,12m+11+i), T(0,36m+21-3i,24m+16-i) \mid$$
$$i = k_2, k_2+1, \ldots, 2m-3 \text{ (omit these triples if } k_2 = 2m-2)\}$$

$\cup\{C(0,8+3i,16m+16+2i), C(0,36m+19-3i,20m+11-2i) \mid$
$i=0,1,\ldots,k_3-1$ (omit these triples if $k_3=0$)$\}$
$\cup\{T(0,8+3i,16m+16+2i), T(0,36m+19-3i,20m+11-2i) \mid$
$i=k_3,k_3+1,\ldots,m-2$ (omit these triples if $k_3=m-1$)$\}$
$\cup\{C(0,6m-4-3i,20m+8-2i), C(0,30m+31+3i,16m+19+2i) \mid$
$i=0,1,\ldots,k_4-1$ (omit these triples if $k_4=0$ or if $m=2$)$\}$
$\cup\{T(0,6m-4-3i,20m+8-2i), T(0,30m+31+3i,16m+19+2i) \mid$
$i=k_4,k_4+1,\ldots,m-3$ (omit these triples if $k_4=m-2$or if $m=2$)$\}$
$\cup\{C(0,6m+2,12m+7), C(0,30m+25,24m+20)$ (omit these triples if $k\leqslant 6m-4$)$\}$
$\cup\{T(0,6m+2,12m+7), T(0,30m+25,24m+20)$ (omit these triples if $k>6m-4$)$\}$
$\cup\{C(0,6m+3,20m+13), C(0,30m+24,16m+14)$ (omit these triples if $k\leqslant 6m-3$)$\}$
$\cup\{T(0,6m+3,20m+13), T(0,30m+24,16m+14)$ (omit these triples if $k>6m-3$)$\}$
$\cup\{C(0,3,16m+12), C(0,36m+24,20m+15)$ (omit these triples if $k\leqslant 6m-2$)$\}$
$\cup\{T(0,3,16m+12), T(0,36m+24,20m+15)$ (omit these triples if $k>6m-2$)$\}$
$\cup\{C(0,2,16m+13), C(0,36m+25,20m+14)$ (omit these triples if $k\leqslant 6m-1\}$
$\cup\{T(0,2,16m+13), T(0,36m+25,20m+14)$ (omit these triples if $k>6m-1$)$\}$
$\cup\{C(0,5,16m+15), C(0,36m+22,20m+12)$ (omit these triples if $k\leqslant 6m$)$\}$
$\cup\{T(0,5,16m+15), T(0,36m+22,20m+12)$ (omit these triples if $k>6m$)$\}$
$\cup\{C(0,8m+8,8m+7), C(0,14m+9,28m+20)$ (omit these triples if $k\leqslant 6m+1$)$\}$
$\cup\{T(0,8m+8,8m+7), T(0,14m+9,28m+20)$ (omit these triples if $k=6m+2$)$\}$.

Case 7: Suppose $v \equiv 33 \pmod{36}$ and $c \equiv 5v/3 \pmod{2v}$, say $v = 36m+33$, $m \geqslant 0$, and $c = 2kv + 5v/3$. In this case we have $0 \leqslant k \leqslant 6m+3$. Define

$$k_1 = \begin{cases} k & \text{if } k \leqslant 2m, \\ 2m & \text{if } k > 2m, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant 2m, \\ k-2m & \text{if } 2m < k \leqslant 4m+1, \\ 2m+1 & \text{if } k > 4m+1, \end{cases}$$

$$k_3 = \begin{cases} 0 & \text{if } k \leqslant 4m+1, \\ k-4m-1 & \text{if } 4m+1 < k \leqslant 5m+2, \\ m+1 & \text{if } k > 5m+2, \end{cases}$$

$$k_4 = \begin{cases} 0 & \text{if } k \leqslant 5m+2, \\ k-5m-2 & \text{if } 5m+2 < k \leqslant 6m+2, \\ m & \text{if } k = 6m+3. \end{cases}$$

Consider the set

$$\begin{aligned}
&\{C(0,6m+5,1), C(0,12m+11,24m+22), C(0,24m+22,12m+11)\} \\
&\quad \cup \{T(0,1,28m+26), T(0,6m+4,30m+27), T(0,6m+6,14m+13)\} \\
&\quad \cup \{C(0,4+3i,8m+10+2i), C(0,36m+29-3i,28m+23-2i) \mid \\
&\qquad i = 0,1,\ldots,k_1-1 \text{ (omit these triples if } k_1 = 0 \text{ or if } m=0)\} \\
&\quad \cup \{T(0,4+3i,8m+10+2i), T(0,36m+29-3i,28m+23-2i) \mid \\
&\qquad i = k_1, i = k_1, k_1+1,\ldots,2m-1 \text{ (omit these triples if } k_1 = 2m \text{ or if } m=0)\} \\
&\quad \cup \{C(0,3+3i,12m+12+i), C(0,36m+30-3i,24m+21-i) \mid \\
&\qquad i = 0,1,\ldots,k_2-1 \text{ (omit these triples if } k_2 = 0)\} \\
&\quad \cup \{T(0,3+3i,12m+12+i), T(0,36m+30-3i,24m+21-i) \mid \\
&\qquad i = k_2, i = k_2, k_2+1,\ldots,2m \text{ (omit these triples if } k_2 = 2m+1)\} \\
&\quad \cup \{C(0,2+3i,16m+16+2i), C(0,36m+31-3i,20m+17-2i) \mid \\
&\qquad i = 0,1,\ldots,k_3-1 \text{ (omit these triples if } k_3 = 0)\} \\
&\quad \cup \{T(0,2+3i,16m+16+2i), T(0,36m+31-3i,20m+17-2i) \mid \\
&\qquad i = k_3, k_3+1,\ldots,m \text{ (omit these triples if } k_3 = m+1)\} \\
&\quad \cup \{C(0,6m+2-3i,20m+16-2i), C(0,30m+31+3i,16m+17+2i) \mid \\
&\qquad i = 0,1,\ldots,k_4-1 \text{ (omit these triples if } k_4 = 0 \text{ or if } m=0)\} \\
&\quad \cup \{T(0,6m+2-3i,20m+16-2i), T(0,30m+31+3i,16m+17+2i) \mid \\
&\qquad i = k_4, k_4+1,\ldots,m-1 \text{ (omit these triples if } k_4 = m \text{ or if } m=0)\}
\end{aligned}$$

$\cup \{C(0,8m+8,20m+18), C(0,22m+20,16m+15)$ (omit these triples if $k<6m+3)\}$

$\cup \{T(0,8m+8,20m+18), T(0,22m+20,16m+15)$ (omit these triples if $k=6m+3)\}$.

In each case, the given set is a set of base triples for a cyclic c-HTS(v) under the permutation $\pi=(0,1,\ldots,v-1)$. □

Lemma 2.7. *If* $v\equiv 0 \pmod{12}$, *then there exists a cyclic* c-HTS(v) *for all* $c\in B_v$ *such that* $c\equiv 2v/3 \pmod{v}$, *except for* $c=b_v$.

Proof

Case 1: Suppose $v\equiv 0 \pmod{12}$ and $c\equiv 2v/3 \pmod{2v}$, say $v=12m$, and $c=2kv+2v/3$. Define

$$k_1=\begin{cases} k & \text{if } k\leqslant m,\\ m & \text{if } k>m,\end{cases}$$

$$k_2=\begin{cases} 0 & \text{if } k\leqslant m,\\ k-m & \text{if } k>m.\end{cases}$$

Consider the set

$\{C(0,4m,8m), C(0,8m,4m), T(0,5m,3m)\}$

$\cup \{C(0,6m-i,1+i), C(0,2m+i,12m-1-i) \mid i=0,1,\ldots,k_1-1$ (omit these triples if $k_1=0)\}$

$\cup \{T(0,6m-i,1+i), T(0,2m+i,12m-1-i) \mid i=k_1,k_1+1,\ldots,m-1$ (omit these triples if $k_1=m)\}$

$\cup \{C(0,5m-1-i,m+1+i), C(0,3m+1+i,11m-1-i) \mid i=0,1,\ldots,k_2-1$ (omit these triples if $k_2=0$ or if $m=1)\}$

$\cup \{T(0,5m-1-i,m+1+i), T(0,3m+1+i,11m-1-i) \mid i=k_2,k_2+1,\ldots,m-2$ (omit these triples if $k_2=m-1$ or if $m=1)\}$.

Case 2: Suppose $v\equiv 0 \pmod{12}$ and $c\equiv 5v/3 \pmod{2v}$, say $v=12m$, and $c=2kv+5v/3$. In the case of $c=b_v$, then a cyclic c-HTS(v) would also be a cyclic MTS(v), which does not exist. So we consider only $0\leqslant k\leqslant 2m-2$. If $m=1$ then consider the set

$$\{C(0,4,8), C(0,8,4), C(0,6,1), T(0,10,1), T(0,9,2)\}.$$

Now suppose $m>1$. Define

$$k_1=\begin{cases} k & \text{if } k\leqslant m-1,\\ m-1 & \text{if } k>m-1,\end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } k \leqslant m-1, \\ k-m+1 & \text{if } k > m-1. \end{cases}$$

Consider the set

$$\begin{aligned}
&\{C(0,4m,8m), C(0,8m,4m), C(0,5m+1,m), T(0,10m,m), T(0,10m-1,2m)\} \\
&\quad \cup \{C(0,2m+1+i,12m-1-i), C(0,6m+1+2i,1+i) \mid i=0,1,\ldots,k_1-1 \\
&\qquad \text{(omit these triples if } k_1=0)\} \\
&\quad \cup \{T(0,2m+1+i,12m-1-i), T(0,6m+1+2i,1+i) \mid i=k_1,k_1+1,\ldots,m-2 \\
&\qquad \text{(omit these triples if } k_1=m-1)\} \\
&\quad \cup \{C(0,3m+1+i,11m-1-i), C(0,8m+1+2i,m+1+i) \mid i=0,1,\ldots,k_2-1 \\
&\qquad \text{(omit these triples if } k_2=0)\} \\
&\quad \cup \{T(0,3m+1+i,11m-1-i), T(0,8m+1+2i,m+1+i) \mid i=k_2,k_2+1,\ldots,m-2 \\
&\qquad \text{(omit these triples if } k_2=m-1)\}.
\end{aligned}$$

In both cases, the given set is a set of base triples for a cyclic c-HTS(v) under the permutation $\pi = (0,1,\ldots,v-1)$. □

The results of this section combine to give us

Theorem 2.1. *A cyclic c-HTS(v) exists if and only if*
(i) $v \equiv 1 \pmod 6$ *and* $t \in \{0,2v,3v,4v,\ldots,v(v-1)/3\}$, *or*
(ii) $v \equiv 4 \pmod{12}$ *and* $t \in \{v,2v,3v,\ldots,v(v-1)/3\}$, *or*
(iii) $v \equiv 3 \pmod 6, v \neq 9$ *and* $t \in \{0,2v,3v,4v,\ldots,v(v-3)/3\}$, *or*
(iv) $v \equiv 0 \pmod{12}$ *and* $t \in \{v,2v,3v,\ldots,v(v-3)/3\}$
where $t = b_v - c = v(v-1)/3 - c$.

3. Rotational hybrid triple systems

In this section, we construct rotational c-HTS(v)s based on the vertex set $\{\infty\} \cup \mathbf{Z}_{v-1}$ and admitting the automorphism $\rho = (\infty)(0,1,\ldots,v-2)$. We have the following necessary conditions:

Lemma 3.1. *If a rotational c-HTS(v) exists, then $v \equiv 0 \pmod 3$ and $c \equiv 0 \pmod{v-1}$ or $v \equiv 1 \pmod 3$ and $c \equiv (v-1)/3 \pmod{v-1}$. Also, if $v \equiv 0 \pmod 6$ or if $v = 10$, then $c \neq v(v-1)/3$.*

Proof. The orbit of any transitive triple under ρ is of length $v-1$. Therefore, $t = v(v-1)/3 - c \equiv 0 \pmod{v-1}$. The result follows from a counting argument, along with the fact that a rotational MTS(v) exists if and only if $v \equiv 1$, 3 or $4 \pmod 6$, $v \neq 10$. □

We now show that these necessary conditions are sufficient.

Lemma 3.2. *If* $v \equiv 4 \pmod 6$, *then there exists a rotational* c-HTS(v) *for all* $c \in B_v$ *such that* $c \equiv (v-1)/3 \pmod{v-1}$, *except for the case* $v = 10$ *and* $c = 30$.

Proof. First, suppose $v = 10$. If $c = 3$, consider the set $\{T(0,\infty,7), C(0,3,6), T(0,1,5), T(0,2,8)\}$. If $c = 12$, consider the set $\{C(0,\infty,2), C(0,3,6), T(0,1,5), T(0,2,8)\}$. If $c = 21$, consider the set $\{C(0,\infty,7), C(0,6,3), T(0,1,5), C(0,3,1)\}$. In each case, the given set is a set of base triples for a rotational c-HTS(10) under the permutation $\rho = (\infty)(0,1,\ldots,8)$.

Now suppose $v \equiv 4 \pmod 6$, $v \neq 10$. Then by case 1 of Lemma 2.6, there exists a cyclic c-HTS($v-1$) for all $c \equiv 2(v-1)/3 \pmod{2(v-1)}$. Let β^* be a set of base triples for such a design. Then β^* contains two triples, say b_1 and b_2, such that the orbits of b_1 and b_2 are each of length $(v-1)/3$ (notice then that b_1 is in the orbit of the block $C(0,(v-1)/3,2(v-1)/3)$ and b_2 is in the orbit of the block $C(2(v-1)/3,(v-1)/3,0)$). Consider the sets

$$\beta_1 = \{C(0,\infty,(v-1)/3), C(0,(v-1)/3,2(v-1)/3)\} \cup \beta^* \setminus \{b_1,b_2\}$$

and

$$\beta_2 = \{T(0,\infty,2(v-1)/3), C(0,(v-1)/3,2(v-1)/3)\} \cup \beta^* \setminus \{b_1,b_2\}.$$

Then β_1 is a set of base triples for a rotational c-HTS(v) where $c \equiv 4(v-1)/3 \pmod{2(v-1)}$ and β_2 is a set of base triples for a rotational c-HTS(v) where $c \equiv (v-1)/3 \pmod{2(v-1)}$. Therefore, there exists a rotational c-HTS(v) for all $c \equiv (v-1)/3 \pmod{v-1}$. □

Lemma 3.3. *If* $v \equiv 1 \pmod 6$, *then there exists a rotational* c-HTS(v) *for all* $c \in B_v$ *such that* $c \equiv (v-1)/3 \pmod{v-1}$.

Proof. We consider two cases.

Case 1: First, if $c = v(v-1)/3$ then a rotational c-HTS(v) is also a rotational MTS(v) and such a system is known to exist [1]. Now suppose $v \equiv 1 \pmod{12}$ and $c \equiv (v-1)/3 \pmod{v-1}$, say $v = 12m+1$ and $c = (v-1)/3 + k(v-1)$ where $k < (v-1)/3$. Define

$$k_1 = \begin{cases} \lfloor k/2 \rfloor & \text{if } \lfloor k/2 \rfloor \leqslant m-1, \\ m-1 & \text{if } \lfloor k/2 \rfloor > m-1, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } \lfloor k/2 \rfloor \leqslant m-1, \\ \lfloor k/2 \rfloor - m + 1 & \text{if } m-1 < \lfloor k/2 \rfloor \leqslant 2m-2, \\ m-1 & \text{if } \lfloor k/2 \rfloor > 2m-2. \end{cases}$$

Consider the set

$$\{T(0,\infty,8m) \text{ (omit this triple if } k \text{ is odd)}\}$$
$$\cup\{C(0,\infty,4m) \text{ (omit this triple if } k \text{ is even)}\} \cup \{T(0,m,3m), C(0,4m,8m)\}$$
$$\cup\{C(0,10m,8m-1), C(0,5m+1,4m+1) \text{ (omit these triples if } k<4m-2)\}$$
$$\cup\{T(0,10m,8m-1), T(0,5m+1,4m+1) \text{ (omit these triples if } k\geqslant 4m-2)\}$$
$$\cup\{C(0,2m+1+i,12m-1-i), C(0,6m-i,1+i) \mid i=0,1,\ldots,k_1-1 \text{ (omit these triples if } k_1=0 \text{ or if } m=1)\}$$
$$\cup\{T(0,2m+1+i,12m-1-i), T(0,6m-i,1+i) \mid i=k_1,k_1+1,\ldots,m-2 \text{ (omit these triples if } k_1=m-1 \text{ or if } m=1)\}$$
$$\cup\{C(0,3m+1+i,11m-1-i), C(0,5m-i,m+1+i) \mid i=0,1,\ldots,k_2-1 \text{ (omit these triples if } k_2=0 \text{ or if } m=1)\}$$
$$\cup\{T(0,3m+1+i,11m-1-i), T(0,5m-i,m+1+i) \mid i=k_2,k_2+1,\ldots,m-2 \text{ (omit these triples if } k_2=m-1 \text{ or if } m=1)\}.$$

Case 2: First, if $c = v(v-1)/3$ then a rotational c-HTS(v) is also a rotational MTS(v) and such a system is known to exist [1]. Now suppose $v \equiv 7 \pmod{12}$ and $c \equiv (v-1)/3 \pmod{v-1}$, say $v = 12m+7$ and $c = (v-1)/3 + k(v-1)$ where $k < (v-1)/3$. Define

$$k_1 = \begin{cases} \lfloor k/2 \rfloor & \text{if } \lfloor k/2 \rfloor \leqslant m, \\ m & \text{if } \lfloor k/2 \rfloor > m, \end{cases}$$

$$k_2 = \begin{cases} 0 & \text{if } \lfloor k/2 \rfloor \leqslant m, \\ \lfloor k/2 \rfloor - m & \text{if } m < \lfloor k/2 \rfloor \leqslant 2m-1, \\ m-1 & \text{if } \lfloor k/2 \rfloor > 2m-1. \end{cases}$$

Consider the set

$$\{T(0,\infty,2m+1) \text{ (omit this triple if } k \text{ is odd)}\}$$
$$\cup\{C(0,\infty,10m+5) \text{ (omit this triple if } k \text{ is even)}\}$$
$$\cup\{T(0,11m+5,4m+2), C(0,8m+4,4m+2)\}$$
$$\cup\{C(0,m+1,4m+3), C(0,10m+4,8m+3) \text{ (omit these triples if } k<4m \text{ or if } m=0)\}$$
$$\cup\{T(0,m+1,4m+3), T(0,10m+4,8m+3) \text{ (omit these triples if } k\geqslant 4m \text{ or if } m=0)\}$$
$$\cup\{C(0,2m+2+i,12m+5-i), C(0,6m+3-i,1+i) \mid i=0,1,\ldots,k_1-1 \text{ (omit these triples if } k_1=0 \text{ or if } m=0)\}$$

$$\cup\{T(0,2m+2+i,12m+5-i),T(0,6m+3-i,1+i) \mid i=k_1,k_1+1,\dots,m-1$$
$$\text{(omit these triples if } k_1=m \text{ or if } m=0)\}$$
$$\cup\{C(0,3m+3+i,11m+4-i),C(0,5m+2-i,m+2+i) \mid i=0,1,\dots,k_2-1$$
$$\text{(omit these triples if } k_2=0 \text{ or if } m\leqslant 1)\}$$
$$\cup\{T(0,3m+3+i,11m+4-i),T(0,5m+2-i,m+2+i) \mid i=k_2,k_2+1,\dots,m-2$$
$$\text{(omit these triples if } k_2=m-1 \text{ or if } m\leqslant 1)\}.$$

In both cases, the given set is a set of base triples for a rotational c-HTS(v) under the permutation $\rho=(\infty)(0,1,\cdots,v-2)$. □

Lemma 3.4. *If* $v\equiv 0 \pmod 3$, *then there exists a rotational* c-HTS(v) *for all* $c\in B_v$ *such that* $c\equiv 0 \pmod{v-1}$, *except the case of* $v\equiv 0 \pmod 6$ *and* $c=v(v-1)/3$.

Proof. We consider two cases.

Case 1: Suppose $v\equiv 0 \pmod 6$ and $c\equiv 0 \pmod{v-1}$, say $v=6m$ and $c=k(v-1)$. Consider the set

$$\{T(0,\infty,2m) \text{ (omit this triple if } k \text{ is odd)}\}$$
$$\cup\{C(0,\infty,4m-1) \text{ (omit this triple if } k \text{ is even)}\}\cup\{T(0,3m,5m-1)\}$$
$$\cup\{C(0,m+i,6m-2-i),C(0,3m-1-i,1+i) \mid i=0,1,\dots,\lfloor k/2\rfloor-1$$
$$\text{(omit these triples if } k=0 \text{ or } 1, \text{or if } m=1)\}$$
$$\cup\{T(0,m+i,6m-2-i),T(0,3m-1-i,1+i) \mid i=\lfloor k/2\rfloor,\lfloor k/2\rfloor+1,\dots,m-2$$
$$\text{(omit these triples if } m=1)\}.$$

Case 2: Suppose $v\equiv 3 \pmod 6$ and $c\equiv 0 \pmod{v-1}$, say $v=6m-3$, $m\geqslant 1$, and $c=k(v-1)$. Consider the set

$$\{T(0,\infty,3m-2) \text{ (omit this triple if } k \text{ is odd)}\}$$
$$\cup\{C(0,\infty,3m-2) \text{ (omit this triple if } k \text{ is even)}\}$$
$$\cup\{C(0,m+i,6m-5-i),C(0,3m+2i,1+i) \mid i=0,1,\dots,\lfloor k/2\rfloor-1$$
$$\text{(omit these triples if } k=0 \text{ or } 1, \text{or if } m=1)\}$$
$$\cup\{T(0,m+i,6m-5-i),T(0,3m+2i,1+i) \mid i=\lfloor k/2\rfloor,\lfloor k/2\rfloor+1,\dots,m-2$$
$$\text{(omit these triples if } k=2m-1 \text{ or } m=1)\}.$$

In both cases, the given set is a set of base triples for a rotational c-HTS(v) under the permutation $\rho=(\infty)(0,1,\dots,v-2)$. □

The results of this section combine to give us

Theorem 3.1. *A rotational c-*HTS(v) *exists if and only if*

(i) $v = 10$ *and* $t \in \{9, 18, 27\}$, *or*

(ii) $v \equiv 1 \pmod 3), v \neq 10$, *and* $t \in \{0, (v-1), 2(v-1), 3(v-1), \ldots, (v-1)^2/3$, *or*

(iii) $v \equiv 0 \pmod 6$ *and* $t \in \{(v-1), 2(v-1), 3(v-1), \ldots, v(v-1)/3\}$, *or*

(iv) $v \equiv 3 \pmod 6$ *and* $t \in \{0, (v-1), 2(v-1), 3(v-1), \ldots, v(v-1)/3\}$

where $t = b_v - c = v(v-1)/3 - c$.

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