Optimal Packings and Coverings of the Complete Directed Graph with 3-Circuits and with Transitive Triples

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Abstract. Maximal packings and minimal coverings of the complete directed graph with isomorphic copies of the directed graph d are studied in the cases of d being either of the two orientations of a 3−cycle. Necessary conditions are given which are shown to be sufficient through direct constructions.

1 Introduction

A *maximal packing* of a simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $E(g_i) \bigcap E(g_j) = \emptyset$ if $i \neq j$, $\bigcup_{i=1}^{n} g_i \subset G$, and

$$\left| E(G) \setminus \bigcup_{i=1}^{n} E(g_i) \right|$$

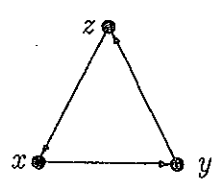
is minimal, where $V(G)$ is the vertex set of graph G and $E(G)$ is the edge set of graph G. Packings of the complete graph on v vertices, K_v, with graph g have been studied for g a 3-cycle [8], g a 4-cycle [9], $g = K_4$ [1], and g a 6-cycle [4,5].

A *minimal covering* of a simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $G \subset \bigcup_{i=1}^{n} g_i$, and

$$\left| \bigcup_{i=1}^{n} E(g_i) \setminus E(G) \right|$$

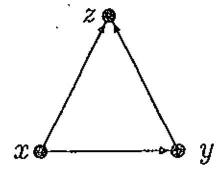
is minimal (the graph $\bigcup_{i=1}^{n} g_i$ may not be simple and $\bigcup_{i=1}^{n} E(g_i)$ may be a multiset). Coverings of K_v with graph g have been studied for g a 3-cycle [2], g a 4-cycle [9], and g a 6-cycle [6].

We define a *maximal packing* and a *minimal covering* of a simple directed graph in a way analogous to the case of undirected graphs. There are two orientations of the 3-cycle: the 3-circuit, C_3, and the transitive triple T. We denote the 3-circuit



[1] Research supported by ETSU under RDC grant #96-029/mjr.

by any cyclic shift of $[x, y, z]_C$ and we denote the transitive triple



by $[x, y, z]_T$. Denote the complete directed graph on v vertices as D_v. The purpose of this paper is to give necessary conditions for packings and coverings of D_v with isomorphic copies of C_3 and T. These necessary conditions are then shown to be sufficient through direct (as opposed to recursive) constructions.

2 The Packing Problem

If $\{d_1, d_2, \ldots, d_n\}$ is a packing of D_v with copies of d, then following the terminology of Kennedy [4] we define the directed graph $L = D_v - \bigcup_{i=1}^{n} d_i$ as the *leave* of the packing. That is, the arc set of L is $A(L) = A(D_v) \setminus \bigcup_{i=1}^{n} A(d_i)$ and the vertex set of L is induced by $A(L)$ (therefore L has no isolated vertices). A maximal packing of D_v with copies of d will therefore make $|A(L)|$ minimal. In the event that $|A(L)| = 0$, it is said that D_v can be *decomposed* into copies of d. A decomposition of D_v into copies of C_3 exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$ and such a decomposition is called a *Mendelsohn triple system* [7]. A decomposition of D_v into copies of T exists if and only if $v \equiv 0$ or $1 \pmod 3$ and such a decomposition is called a *directed triple system* [3]. Therefore, we need only consider the problem of packing D_v with copies of T (or copies of C_3) when $v \equiv 2 \pmod 3$ (and $v = 6$).

We first consider the question of packing D_v with copies of T. For brevity, we no longer make a distinction between graphs being "isomorphic" and "equal."

Theorem 2.1 *A maximal packing of D_v with copies of the transitive triple T and leave L satisfies:*

1. $|A(L)| = 0$ *if* $v \equiv 0$ *or* $1 \pmod 3$, *or*

2. $|A(L)| = 2$ *and* $L = C_2$ *if* $v \equiv 2 \pmod 3$.

Proof. With $v \equiv 2 \pmod 3$, $|A(D_v)| = v(v-1) \equiv 2 \pmod 3$ and if we can demonstrate a packing where $|A(L)| = 2$, then it certainly must be maximal.
Case 1. If $v \equiv 2 \pmod{12}$, say $v = 12t + 2$, then consider the set of triples:

$$\{[0, 3t - i, 3t + 1 + i]_T \mid i = 0, 1, \ldots, t-1\} \cup$$

$$\{[0, 5t - i, 5t + 2 + i]_T \mid i = 0, 1, \ldots, t-1\} \cup$$

$$\{[0, 7t + 2 + i, 7t - i]_T \mid i = 0, 1, \ldots, t-2\} \cup$$

$$\{[0, 9t + 1 + i, 9t - i]_T \mid i = 0, 1, \ldots, t-1\} \cup \{[0, x, 5t + 1]_T, [0, y, 7t + 1]_T\}.$$

Case 2. If $v \equiv 5 \pmod{12}$, say $v = 12t + 5$, then consider the set of triples:

$$\{[0, 3t - i, 3t + 1 + i]_T \mid i = 0, 1, \ldots, t - 1\} \cup$$
$$\{[0, 5t - i, 5t + 2 + i]_T \mid i = 0, 1, \ldots, t - 1\} \cup$$
$$\{[0, 7t + 3 + i, 7t + 1 - i]_T \mid i = 0, 1, \ldots, t - 1\} \cup$$
$$\{[0, 9t + 3 + i, 9t + 2 - i]_T \mid i = 0, 1, \ldots, t - 1\} \cup \{[0, x, 5t + 1]_T, [0, y, 7t + 2]_T\}.$$

Case 3. If $v \equiv 8 \pmod{12}$, say $v = 12t + 8$, then consider the set of triples:

$$\{[0, 3t + 2 - i, 3t + 3 + i]_T \mid i = 0, 1, \ldots, t\} \cup$$
$$\{[0, 5t + 3 - i, 5t + 5 + i]_T \mid i = 0, 1, \ldots, t - 1\} \cup$$
$$\{[0, 7t + 6 + i, 7t + 4 - i]_T \mid i = 0, 1, \ldots, t - 1\} \cup$$
$$\{[0, 9t + 6 + i, 9t + 5 - i]_T \mid i = 0, 1, \ldots, t - 1\} \cup \{[0, x, 5t + 4]_T, [0, y, 7t + 5]_T\}.$$

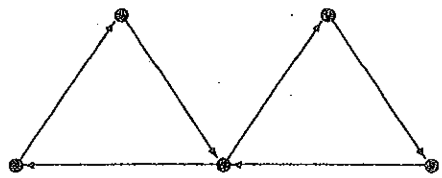
Case 4. If $v \equiv 11 \pmod{12}$, say $v = 12t + 11$, then consider the set of triples:

$$\{[0, 3t + 2 - i, 3t + 3 + i]_T \mid i = 0, 1, \ldots, t\} \cup$$
$$\{[0, 5t + 3 - i, 5t + 5 + i]_T \mid i = 0, 1, \ldots, t - 1\} \cup$$
$$\{[0, 7t + 6 + i, 7t + 4 - i]_T \mid i = 0, 1, \ldots, t - 1\} \cup$$
$$\{[0, 9t + 7 + i, 9t + 6 - i]_T \mid i = 0, 1, \ldots, t\} \cup \{[0, x, 5t + 4]_T, [0, y, 7t + 5]_T\}.$$

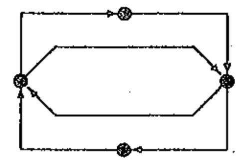
In each case, the given set of triples along with their images under the powers of the permutation $(x)(y)(0, 1, \ldots, v - 3)$ form a packing of D_v, where $V(D_v) = \{x, y, 0, 1, \ldots, v - 3\}$, with copies of C_3 and leave $L = C_2$, where $A(L) = \{(x, y), (y, x)\}$.

Finally, we note that the total-degree (i.e. the in-degree plus the out-degree) of each vertex of D_v is $2(v - 1)$ and the total degree of each vertex of T is 2. So any packing of D_v with copies of T will have a leave L with each vertex of even total-degree. Therefore, an optimal packing must have $L = C_2$. ∎

We now consider packing D_v with copies of C_3. Since each vertex of D_v has in-degree equal to out-degree and each vertex of C_3 has in-degree equal to out-degree, it must be that each vertex of a leave also has this property. The directed graph D_6 has 30 arcs. Therefore if L is a leave for a packing of D_6 with copies of C_3, then $|A(L)| \equiv 0 \pmod 3$. Since D_6 cannot be decomposed into copies of C_3, $|A(L)| \neq 0$. If $|A(L)| = 3$, then it would be necessary for $L = C_3$, a contradiction. So if a packing of D_6 with copies of C_3 can be demonstrated with $|A(L)| = 6$, the packing will be maximal. There are nine directed graphs with six arcs in which in-degree equals out-degree for each vertex:



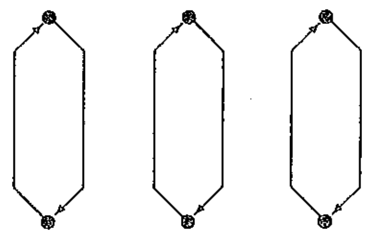
S_1



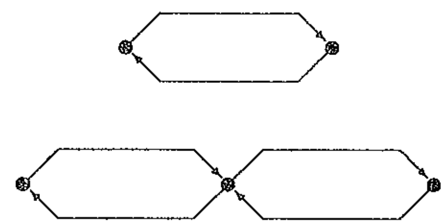
S_2

$S_3 = D_3$

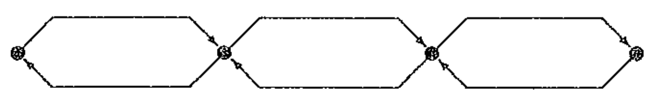
$S_4 = C_6$



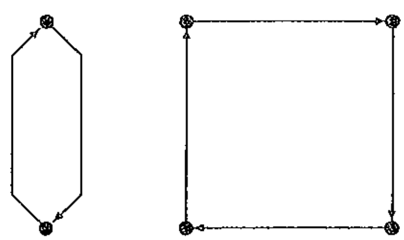
S_5



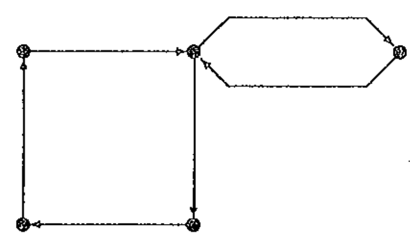
S_6



S_7



S_8



S_9

Since S_1, S_2, and S_3 can each be decomposed into copies of C_3, $L \notin \{S_1, S_2, S_3\}$. Suppose that $L = S_6$ and $A(L) = \{(0,1),(1,0),(1,2),(2,1),(4,5),(5,4)\}$ where $V(D_6) = \{0,1,\ldots,5\}$. Then $[1,5,3]_C$ and $[1,4,3]_C$ must both be blocks of the packing of D_6 with copies of C_3 and leave $L = S_6$. However, this is clearly a contradiction since both triples contain the arc $(3,1)$. Therefore $L \neq S_6$. In the following result, we show that L may be any of the directed graphs S_4, S_5, S_7, S_8, S_9.

Lemma 2.1 *A maximal packing of D_6 with copies of the 3−circuit C_3 and leave L satisfies $|A(L)| = 6$ and L may be any element of the set $\{S_4, S_5, S_7, S_8, S_9\}$.*

Proof. Let $V(D_6) = \{0,1,\ldots,5\}$.
<u>Case 1.</u> $L = S_4$.

Consider the set $\{[0,1,3]_C, [0,2,5]_C, [0,3,4]_C, [0,4,2]_C, [1,2,4]_C, [1,4,5]_C, [1,5,3]_C, [2,3,5]_C\}$. This is a packing of D_6 with copies of C_3 and leave $L = S_4$ where $A(L) = \{(5,4), (4,3), (3,2), (2,1), (1,0), (0,5)\}$.

Case 2. $L = S_5$.

Consider the set $\{[0,1,3]_C, [0,2,4]_C, [0,3,2]_C, [0,4,1]_C, [1,4,5]_C, [1,5,3]_C, [2,3,5]_C, [2,5,4]_C\}$. This is a packing of D_6 with copies of C_3 and leave $L = S_5$ where $A(L) = \{(1,2), (2,1), (3,4), (4,3), (0,5), (5,0)\}$.

Case 3. $L = S_7$.

Consider the set $\{[0,2,4]_C, [0,3,5]_C, [0,4,3]_C, [0,5,2]_C, [1,3,4]_C, [1,4,5]_C, [1,5,3]_C, [2,5,4]_C\}$. This is a packing of D_6 with copies of C_3 and leave $L = S_5$ where $A(L) = \{(0,1), (1,0), (1,2), (2,1), (2,3), (3,2)\}$.

Case 4. $L = S_8$.

Consider the set $\{[1,3,2]_C, [2,3,4]_C, [4,3,5]_C, [5,3,1]_C, [0,1,4]_C, [0,2,5]_C, [0,4,1]_C, [0,5,2]_C\}$. This is a packing of D_6 with copies of C_3 and leave $L = S_5$ where $A(L) = \{(0,3), (3,0), (1,2), (2,4), (4,5), (5,1)\}$.

Case 5. $L = S_9$.

Consider the set $\{[0,2,4]_C, [0,3,5]_C, [0,5,1]_C, [0,4,3]_C, [4,5,3]_C, [4,2,5]_C, [1,2,3]_C, [1,5,2]_C\}$. This is a packing of D_6 with copies of C_3 and leave $L = S_5$ where $A(L) = \{(0,1), (1,4), (4,1), (1,3), (3,2), (2,0)\}$. ∎

Theorem 2.2 *A maximal packing of D_v, where $v \neq 6$, with copies of the 3−circuit C_3 and leave L satisfies:*

1. $|A(L)| = 0$ *if* $v \equiv 0$ *or* $1 \ (mod\ 3)$, $v \neq 6$, *or*
2. $|A(L)| = 2$ *and* $L = C_2$ *if* $v \equiv 2 \ (mod\ 3)$.

Proof. The arguments of Theorem 2.1 again show that the theorem is proved if we can demonstrate a packing of D_v where $v \equiv 2 \pmod 3$ such that $L = C_2$.

Case 1. If $v \equiv 2 \pmod 6$, say $v = 6t+2$, then consider the set of triples:

$$\{[0, 3t+i, 6t-1-i]_C \mid i = 0,1,\ldots,t-1\} \cup$$

$$\{[0, 4t+1+i, t-1-i]_C \mid i = 0,1,\ldots,t-2\} \cup \{[x,0,4t]_C, [y,0,5t]_C\}$$

Case 2. If $v \equiv 5 \pmod 6$, say $v = 6t+5$, then consider the set of triples:

$$\{[0, 2+i, 3t+2-i]_C \mid i = 0,1,\ldots,t-1\} \cup$$

$$\{[0, 4t+4+i, t-i]_C \mid i = 0,1,\ldots,t-2\} \cup \{[x,0,1]_C, [y,0,6t+2]_C\}$$

$$\cup\{[0, 4t+1, 2t]_C (\text{omit if } t = 0)\}.$$

In each case, the given set of triples along with their images under the powers of the permutation $(x)(y)(0,1,\ldots,v-3)$ form a maximal packing of D_v where $V(D_v) = \{x, y, 0, 1, \ldots, v-3\}$ with copies of C_3 and leave $L = C_2$ where $A(L) = \{(x,y),(y,x)\}$. ∎

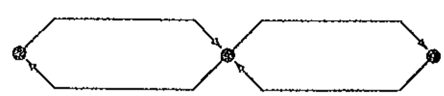
Lemma 2.1 and Theorems 2.1 and 2.2 give necessary and sufficient conditions for the existence of maximal coverings of D_v with copies of T and C_3.

3 The Covering Problem

If $\{d_1, d_2, \ldots, d_n\}$ is a covering of D_v with copies of d, then we define the directed graph $P = \bigcup_{i=1}^{n} d_i - D_v$ as the *padding* of the covering (as with a leave, it is understood that P has no isolated vertices; recall that $\bigcup_{i=1}^{n} d_i$ may not be simple). A minimal covering of D_v with copies of d will therefore make $|A(P)|$ minimal. As discussed in Section 2, if $d = T$ or $d = C_3$ then we need only consider $v \equiv 2 \pmod 3$ and $v = 6$ if $d = C_3$.

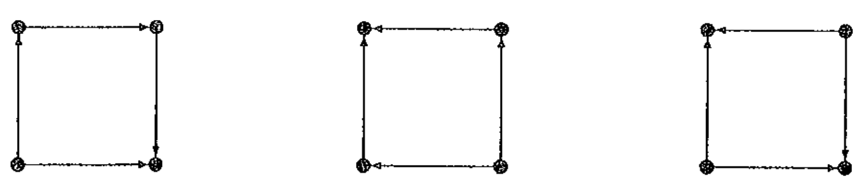
Suppose $v \equiv 2 \pmod 3$ and $d = C_3$. We see that $|A(P)| \equiv 1 \pmod 3$. If $\{d_1, d_2, \ldots, d_n\}$ is a covering of D_v and $|A(P)| = 1$ with $A(P) = \{(x, y)\}$ where $d_1 = [x, y, z]_C$ (with x, y, z distinct), then $\{d_2, d_3, \ldots, d_n\}$ is a packing of D_v with copies of C_3 and leave L where $A(L) = \{(y, z), (z, x)\}$, contradicting Theorem 2.2. A similar argument shows that a covering of D_v with copies of T cannot have a padding P with $|A(P)| = 1$. In this section, we show that a minimal covering of D_v with $v \equiv 2 \pmod 3$ and $d = C_3$ or $d = T$ has a padding P satisfying $|A(P)| = 4$. We give direct constructions of such coverings for each possible form of P.

In Theorem 2.1 it is shown that the total degree of each vertex of the leave of a packing of D_v with C_3 or T is even. An analogous argument shows that the total degree of each vertex of the padding of a covering of D_v with C_3 or T is even. Therefore, if a padding P satisfies $|A(P)| = 4$ then P is either two disjoint copies of C_2, an orientation of a 4-cycle, or two "osculating" C_2s, which we denote as OC_2:



OC_2

Notice that there are four orientations of the 4-cycle. We denote the 4-circuit as C_4 and the other orientations we denote as:



X $\qquad$ Y $\qquad$ Z

We now show that for a covering of D_v with copies of T where $v \equiv 2 \pmod 3$, each of these paddings is possible.

Theorem 3.1 *A minimal covering of D_v with copies of the transitive triple T and padding P satisfies:*

1. $|A(P)| = 0$ *if* $v \equiv 0$ *or* $1 \pmod 3$, *or*

2. $|A(P)| = 4$ *if* $v \equiv 2 \pmod 3$ *and P may be two disjoint copies of C_2, any orientation of a 4−cycle, or two osculating 2 − circuits OC_2.*

Proof. If $v \equiv 0$ or $1 \pmod 3$, then there exists a decomposition of D_v into copies of T [3]. So suppose that $v \equiv 2 \pmod 3$. Then, as described above, $|A(P)| \geq 4$ and if we can demonstrate a covering with $|A(P)| = 4$ then it must be minimal. Also as described above, the only possible forms for P are those listed in the theorem.

Case 1a. $v = 8$ and P is two disjoint C_2s.

Consider the set of triples $\{[0,5,4]_T, [4,5,0]_T, [0,1,4]_T, [4,1,0]_T, [1,3,5]_T, [5,7,1]_T, [1,6,5]_T, [5,2,1]_T, [7,4,6]_T, [6,4,3]_T, [3,4,2]_T, [2,4,7]_T, [6,0,7]_T, [3,0,6]_T, [2,0,3]_T, [7,0,2]_T, [6,1,2]_T, [2,5,6]_T, [7,5,3]_T, [3,1,7]_T\}$. This set is a minimal covering of D_8 where $V(D_8) = \{0, 1, \ldots, 7\}$ with copies of T and padding P where $A(P) = \{(0,4), (4,0), (1,5), (5,1)\}$.

Case 1b. $v \equiv 2 \pmod 6$, $v \neq 8$, and P is two disjoint C_2s.

Let $v = 6t + 2$ where $t \geq 2$. Consider the set of triples:

$$\{[0, 3t-4-i, 3t-3+i]_T \mid i = 0, 1, \ldots, t-2\} \cup$$

$$\{[0, 5t-7-i, 5t-5+i]_T \mid i = 0, 1, \ldots, t-3\} \cup$$

$$\{[0, a, 5t-6]_T, [0, b, 6t-7]_T, [0, c, 6t-6]_T, [0, d, 6t-5]_T, [0, e, 6t-4]_T\} \cup$$

$$\{[a,b,e]_T, [e,b,a]_T, [e,c,a]_T, [a,d,e]_T, [d,a,c]_T, [c,e,d]_T, [d,b,c]_T, [c,b,d]_T\}.$$

The given set of triples along with their images under the powers of the permutation $(a)(b)(c)(d)(e)(0, 1, \ldots, v-6)$ form a minimal covering of D_v where $V(D_v) = \{a, b, c, d, e, 0, 1, \ldots, v-6\}$ with copies of T and padding P where $A(P) = \{(a,e), (e,a), (c,d), (d,c)\}$.

Case 1c. $v \equiv 5 \pmod 6$ and P is two disjoint C_2s.

Let $v = 6t + 5$. Consider the set of triples:

$$\{[0, 3t-3-i, 3t-2+i]_T \mid i = 0, 1, \ldots, t-2\} \cup$$

$$\{[0, 5t-5-i, 5t-3+i]_T \mid i = 0, 1, \ldots, t-2\} \cup$$

$$\{[0, a, 5t-4]_T, [0, b, 6t-4]_T, [0, c, 6t-3]_T, [0, d, 6t-2]_T, [0, e, 6t-1]_T\} \cup$$

$$\{[a,b,e]_T, [e,b,a]_T, [e,c,a]_T, [a,d,e]_T, [d,a,c]_T, [c,e,d]_T, [d,b,c]_T, [c,b,d]_T\}.$$

As in Case 1b, the result follows.

Case 2. $v \equiv 2 \pmod 3$ and $P = C_4$.

We know that there exists a packing of D_v with copies of T with leave $L = C_2$ by Theorem 2.1. Say $A(L) = \{(x,y), (y,x)\}$. Then let w and z be two vertices of D_v distinct from x and y. We can take this packing of D_v along with $[x,w,y]_T$ and $[y,z,x]_T$ to produce a covering of D_v with padding P and $A(P) = \{(x,w), (w,y), (y,z), (z,x)\}$.

Case 3. $v \equiv 2 \pmod 3$ and $P = X$.

We can take a packing of D_v with copies of T as described in Case 2 and add $[x,w,y]_T$ and $[y,x,z]_T$ to produce a covering of D_v with padding P and $A(P) = \{(x,w), (w,y), (y,z), (x,z)\}$.

Case 4. $v \equiv 2 \pmod 3$ and $P = Y$.

We can take a packing of D_v with copies of T as described in Case 2 and add $[x,y,w]_T$ and $[z,y,x]_T$ to produce a covering of D_v with padding P and $A(P) = \{(x,w), (y,w), (z,y), (z,x)\}$.

Case 5. $v \equiv 2 \pmod 3$ and $P = Z$.

We can take a packing of D_v with copies of T as described in Case 2 and add $[x, y, w]_T$ and $[y, x, z]_T$ to produce a covering of D_v with padding P and $A(P) = \{(x, w), (y, w), (y, z), (x, z)\}$.

Case 6. $v \equiv 2 \pmod 3$ and $P = OC_2$.

Again, as in Case 2 we can find a packing of D_v with copies of T and leave L where $A(L) = \{(x, y), (y, x)\}$. Let z be a vertex of D_v distinct from x and y. We can take the packing along with $[x, z, y]_T$ and $[y, z, x]_T$ to produce a covering of D_v with padding P and $A(P) = \{(x, z), (z, x), (y, z), (z, y)\}$. ∎

We now turn our attention to covering D_v with copies of C_3. Since each vertex of D_v has in-degree equal to out-degree and each vertex of C_3 has in-degree equal to out-degree, it must be that each vertex of a padding also has this property. Therefore, the padding cannot be the orientations of the 4-cycle of X, Y, or Z. We show the other possible paddings are each attained for certain coverings.

Theorem 3.2 *A minimal covering of D_v with copies of the 3−circuit C_3 and padding P satisfies*

1. $|A(P)| = 0$ *if* $v \equiv 0$ *or* $1 \pmod 3$, $v \neq 6$,

2. $|A(P)| = 3$ *and* $P = C_3$ *if* $v = 6$, *or*

3. $|A(P)| = 4$ *if* $v \equiv 2 \pmod 3$ *and P may be two disjoint copies of C_2, a 4−circuit, or two osculating* $2-$*circuits* OC_2.

Proof. First, we consider the case $v = 6$ and $P = C_3$. Since D_6 has 30 arcs and a decomposition of D_6 into copies of C_3 does not exist, a covering of D_6 with copies of C_3 satisfying $|A(P)| = 3$ would be minimal. Also, since each vertex of P must satisfy in-degree equals out-degree, it is necessary that such a P be equal to C_3. Let $V(D_6) = \{0, 1, \ldots, 5\}$. Consider the set $\{[0, 1, 3]_C, [0, 2, 5]_C, , [0, 3, 4]_C, [0, 4, 2]_C, [1, 2, 4]_C, [1, 4, 5]_C, [1, 5, 3]_C, [2, 3, 5]_C, [0, 2, 1]_C, [2, 4, 3]_C, [0, 5, 4]_C\}$. This is a covering with $P = \{(0, 2), (2, 4), (4, 0)\}$.

If $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$, then there exists a decomposition of D_v into copies of C_3 [7]. So we now need only consider $v \equiv 2 \pmod 3$.

Case 1a. $v = 8$ and P is two disjoint C_2s.

Consider the set $\{[0, 5, 4]_C, [0, 4, 5]_C, [0, 1, 4]_C, [0, 4, 1]_C, [1, 5, 2]_C, [1, 5, 7]_C, [1, 3, 5]_C, [1, 6, 5]_C, [[4, 6, 7]_C, [4, 3, 6]_C, [4, 7, 2]_C, [4, 2, 3]_C, [0, 7, 6]_C, [0, 6, 3]_C, [0, 2, 7]_C, [0, 3, 2]_C, [1, 2, 6]_C, [2, 5, 6]_C, [1, 7, 3]_C, [3, 7, 5]_C\}$. This set forms a minimal covering of D_8 where $V(D_8) = \{0, 1, \ldots, 7\}$ with copies of C_3 and padding P where $A(P) = \{(0, 4), (4, 0), (1, 5), (5, 1)\}$.

Case 1b. $v \equiv 2 \pmod 6$, $v \neq 8$, and P is two disjoint C_2s.

Let $v = 6t + 2$, $t \geq 2$, and let

$$J_1 = \{[0, 2 + i, 3t - 1 - i]_C \mid i = 0, 1, \ldots, t - 2\}$$

$$K_1 = \{[0, 4t + i, t - 1 - i]_C \mid i = 0, 1, \ldots, t - 3\}$$

$$L_1 = \{[0, 1, a]_C, [0, 4t - 3, b]_C, [0, 4t - 2, c]_C, [0, 4t - 1, d]_C, [0, 6t - 4, e]_C\}$$

$$M = \{[a,b,e]_C, [a,e,b]_C, [a,e,c]_C, [a,d,e]_C, [a,c,d]_C, [c,e,d]_C, [b,c,d]_C, [b,d,c]_C\}.$$

The set of triples $J_1 \bigcup K_1 \bigcup L_1 \bigcup M$ along with their images under the powers of the permutation $(a)(b)(c)(d)(e)(0,1,\ldots,v-6)$ form a minimal covering of D_v, where $V(D_v) = \{a,b,c,d,e,0,1,\ldots,v-6\}$, with copies of C_3 and padding P where $A(P) = \{(a,e),(e,a), (d,c), (c,d)\}$.
Case 1c. $v \equiv 5 \pmod 6$ and P is two disjoint C_2s.
Let $v = 6t+5$ and let

$$J_2 = \{[0,3t+i,6t-1-i]_C \mid i = 0,1,\ldots,t-2\}$$

$$K_2 = \{[0,4t+1+i,t-1-i]_C \mid i = 0,1,\ldots,t-2\}$$

$$L_2 = \{[0,4t,a]_C, [0,5t,b]_C, [0,4t-1,c]_C, [0,t+1,d]_C, [0,t,e]_C\}.$$

The set of triples $J_2 \bigcup K_2 \bigcup L_2 \bigcup M$, where M is as defined in Case 1b, demonstrates this case, as in Case 1b.
Case 2. $v \equiv 2 \pmod 3$ and $P = C_4$.
We know that there exists a packing of D_v with copies of C_3 with leave $L = C_2$ by Theorem 2.2. Say $A(L) = \{(x,y),(y,x)\}$. Then let w and z be two vertices of D_v distinct from x and y. We can take the packing of D_v along with $[x,y,w]_C$ and $[y,x,z]_C$ to produce a covering of D_v with padding P and $A(P) = \{(x,z),(z,y),(y,w),(w,x)\}$.
Case 3. $v \equiv 2 \pmod 3$ and $P = OC_2$.
As in Case 2 we can find a packing of D_v with copies of C_3 and leave L where $A(L) = \{(x,y),(y,x)\}$. Let z be a vertex of D_v distinct from x and y. We can take the packing along with $[x,z,y]_C$ and $[x,y,z]_C$ to produce a covering of D_v with padding P such that $A(P) = \{(x,z),(z,x),(y,z),(z,y)\}$. ∎

Theorems 3.1 and 3.2 give necessary and sufficient conditions for the existence of maximal packings of D_v with copies of T and C_3, respectively.

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