# Bicyclic Directed Triple Systems

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**Abstract.** A directed triple system of order $v$, denoted $DTS(v)$, is said to be *bicyclic* if it admits an automorphism whose disjoint cyclic decomposition consists of two cycles. In this paper, we give necessary and sufficient conditions for the existence of bicyclic $DTS(v)$s.

## 1. Introduction

A *directed triple system* of order $v$, denoted $DTS(v)$, is a $v$-element set $X$ of points, together with a set $\beta$ of ordered triples of elements of $X$, called *blocks*, such that any ordered pair of points of $X$ occur in exactly one block of $\beta$. We denote by $[x, y, z]$ the block containing the ordered pairs $(x, y)$, $(y, z)$ and $(x, z)$. A $DTS(v)$ exists if and only if $v \equiv 0$ or 1 (mod 3) [6].

An *automorphism* of a $DTS(v)$ is a permutation of $X$ which fixes $\beta$. The *orbit* of a block under an automorphism $\pi$ is the image of the block under the powers of $\pi$. A set of blocks $B$ is said to be a *set of base blocks for a $DTS(v)$ under the permutation* $\pi$ if the orbits of the blocks of $B$ produce the $DTS(v)$ and exactly one block of $B$ occurs in each orbit. A $DTS(v)$ admitting an automorphism consisting of a single cycle is said to be *cyclic*. A cyclic $DTS(v)$ exists if and only if $v \equiv 1$, 4, or 7 (mod 12) [4]. A $DTS(v)$ admitting an automorphism consisting of a fixed point and a cycle of length $v-1$ is said to be *rotational* (or *1-rotational*) and exists if and only if $v \equiv 0$ (mod 3) [2].

These types of automorphism questions have also been addressed for other triple systems. Colbourn [3] proved that if $\pi$ is an automorphism of a two-fold triple system of order $v$ then, under the appropriate necessary conditions, the two-fold triple system can be directed to form a $DTS(v)$ which also admits $\pi$ as an automorphism. A Steiner triple system of order $v$, denoted $STS(v)$, is said to be *bicyclic* if it admits an automorphism consisting of two disjoint cycles. A bicyclic $STS(v)$ admitting an automorphism consisting of a cycle of length $N$ and a cycle of length $M$ (where $N < M$) exists if and only if $N \equiv 1$ or 3 (mod 6), $N \neq 9$, $N \mid M$, and $v = N + M \equiv 1$ or 3 (mod 6) [1, 5]. The purpose of this paper is to present necessary and sufficient conditions for the existence of a bicyclic $DTS(v)$.

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We break this into two general cases. In the first, we assume the lengths of the two cycles composing the automorphism are equal. In the second case, we assume the cycles are of different lengths.

## 2. Automorphism Consists of Two Cycles of Equal Lengths

In this section, we consider bicyclic $DTS(v)$s with vertex set $\mathbf{Z}_N \times \mathbf{Z}_2$, where $N = v/2$, and admitting the automorphism $\pi = (0_0, 1_0, \ldots, (N-1)_0)(0_1, 1_1, \ldots, (N-1)_1)$. We have the following necessary condition:

**Lemma 2.1** *If a bicyclic $DTS(v)$ exists which admits an automorphism $\pi$ consisting of two cycles of the same length, then $v \equiv 4$ (mod 6).*

**Proof.** The number of blocks in a $DTS(v)$ is $v(v-1)/3$. The orbit of each block of such a $DTS(v)$ under $\pi$ has length $v/2$. So $v/2$ divides $v(v-1)/3$. This implies that $v \equiv 1$ (mod 3). Since $v$ is also even, we have $v \equiv 4$ (mod 6). ∎

We now show that the necessary conditions of Lemma 2.1 are sufficient in the next four lemmas. We require the use of two structures. An *(A,n)-system* is a collection of ordered pairs $(a_r, b_r)$ for $r = 1, 2, \ldots, n$ that partition the set $\{1, 2, \ldots, 2n\}$ with the property that $b_r = a_r + r$ for $r = 1, 2, \ldots, n$. Skolem proved that an $(A, n)$–system exists for $n \geq 8$ if and only if $n \equiv 0$ or 1 (mod 4) [8]. The result also holds for $n \in \{1, 4, 5\}$ as well, as is shown by the following:

$(A, 1)$–system: $(1, 2)$

$(A, 4)$–system: $(1, 2), (5, 7), (3, 6), (4, 8)$,

$(A, 5)$–system: $(2, 3), (8, 10), (4, 7), (5, 9), (1, 6)$.

A *(B,n)-system* is a collection of ordered pairs $(a_r, b_r)$ for $r = 1, 2, \ldots, n$ that partition the set $\{1, 2, \ldots, 2n-1, 2n+1\}$ with the property that $b_r = a_r + r$ for $r = 1, 2, \ldots, n$. O'Keefe proved that a $(B, n)$–system exists for $n \geq 6$ if and only if $n \equiv 2$ or 3 (mod 4). [7]. This result holds for $n \in \{2, 3\}$ as well, as is shown by the following:

$(B, 2)$–system: $(1, 2), (3, 5)$,

$(B, 3)$–system: $(1, 2), (3, 5), (4, 7)$.

**Lemma 2.2** *If $N \equiv 5$ (mod 24) then there exists a bicyclic $DTS(2N)$*

*admitting an automorphism $\pi$ which consists of two disjoint cycles each of length $N$.*

**Proof.** Suppose $N = 24t + 5$. Consider the set:

$\{[0_{i+1}, (6t+r)_i, (6t+1-r)_i] \mid r = 1, 2, \ldots, 6t+1 \text{ and } i \in \mathbf{Z}_2\} \bigcup$

$\{[0_{i+1}, (18t+3+r)_i, (18t+3-r)_i] \mid r = 1, 2, \ldots, 6t \text{ and } i \in \mathbf{Z}_2\} \bigcup$

$\{[(6t+2)_0, 0_1, (18t+3)_0], [0_1, (24t+4)_0, (12t+2)_0], [1_0, (12t+3)_0, 0_1]\} \bigcup$

$\{[0_0, r_0, (b_r + 4t)_0] \mid r = 1, 2, \ldots, 4t$ and the $b_r$ are from an $(A, 4t)$-system$\} \bigcup$

$\{[0_1, r_1, (b_r + 4t + 1)_1] \mid r = 1, 2, \ldots, 4t+1$ and the $b_r$ are from an $(A, 4t+1)$-system$\}$.

This is a set of base blocks for a bicyclic $DTS(v)$ under the permutation $\pi$. ∎

**Lemma 2.3** *If $N \equiv 11 \pmod{24}$ then there exists a bicyclic $DTS(2N)$ admitting an automorphism $\pi$ which consists of two disjoint cycles each of length $N$.*

**Proof.** Suppose $N = 24t + 11$. Consider the set:

$\{[0_{i+1}, (6t+2+r)_i, (6t+3-r)_i] \mid r = 1, 2, \ldots, 6t+3 \text{ and } i \in \mathbf{Z}_2\} \bigcup$

$\{[0_{i+1}, (18t+8+r)_i, (18t+8-r)_i] \mid r = 1, 2, \ldots, 6t+1 \text{ and } i \in \mathbf{Z}_2\} \bigcup$

$\{[(12t+5)_0, 1_0, 0_1], [(6t+3)_0, 0_1, (18t+8)_0], [0_1, (12t+6)_0, (24t+10)_0]\} \bigcup$

$\{[0_0, r_0, (b_r + 4t + 1)_0] \mid r = 1, 2, \ldots, 4t+1$ and the $b_r$ are from an $(A, 4t+1)$-system$\} \bigcup$

$\{[0_1, r_1, (b_r+4t+2)_1] \mid r = 1, 2, \ldots, 4t+2$ and the $b_r$ are from a $(B, 4t+2)$-system$\}$.

This is a set of base blocks for a bicyclic $DTS(v)$ under the permutation $\pi$. ∎

**Lemma 2.4** *If $N \equiv 17 \pmod{24}$ then there exists a bicyclic $DTS(2N)$ admitting an automorphism $\pi$ which consists of two disjoint cycles each of length $N$.*

**Proof.** Suppose $N = 24t + 17$. Consider the set:

$\{[0_0,(6t+3+r)_1,(6t+4-r)_1],[0_0,(18t+12+r)_1,(18t+12-r)_1],[0_1,(12t+8+r)_0,(12t+8-r)_0] \mid r=1,2,\ldots,6t+4\} \bigcup$

$\{[0_1,(r-1)_0,(24t+17-r)_0] \mid r=1,2,\ldots,6t+3\} \bigcup$

$\{[(18t+14)_1,(6t+4)_1,0_0],[(12t+9)_1,0_0,(18t+12)_1]\} \bigcup$

$\{[0_0,r_0,(b_r+4t+3)_0] \mid r=1,2,\ldots,4t+3$ and the $b_r$ are from a $(B,4t+3)$-system$\} \bigcup$

$\{[0_1,(6t+2-r)_1,(6t+3+r)_1] \mid r=1,2,\ldots,2t-1\} \bigcup$

$\{[0_1,(10t+3-r)_1,(10t+5+r)_1] \mid r=1,2,\ldots,2t\} \bigcup$

$\{[0_1,(10t+4)_1,(10t+5)_1],[0_1,(12t+6)_1,(12t+8)_1]\} \bigcup$

$\{[0_1,(6t+2)_1,(10t+3)_1]$ (omit if $t=1$)$\}$.

This is a set of base blocks for a bicyclic $DTS(v)$ under the permutation $\pi$. ∎

**Lemma 2.5** *If $N \equiv 23 \pmod{24}$ then there exists a bicyclic $DTS(2N)$ admitting an automorphism $\pi$ which consists of two disjoint cycles each of length $N$.*

**Proof.** Suppose $N = 24t+23$. Consider the set:

$\{[0_0,(6t+5+r)_1,(6t+6-r)_1],[0_0,(18t+17+r)_1,(18t+17-r)_1],[0_1,(18t+15+r)_0,(18t+15-r)_0] \mid r=1,2,\ldots,6t+5\} \bigcup$

$\{[0_1,(6t+3+r)_0,(6t+4-r)_0] \mid r=1,2,\ldots,6t+4\} \bigcup$

$\{[0_0,(12t+11)_0,(12t+11)_1],[(6t+6)_0,0_1,(18t+15)_0]\} \bigcup$

$\{[0_1,(12t+9)_0,(24t+21)_0],[0_1,(12t+8)_0,(24t+22)_0]\} \bigcup$

$\{[0_0,r_0,(b_r+4t+3)_0] \mid r=1,2,\ldots,4t+3$ and the $b_r$ are from a $(B,4t+3)$-system$\} \bigcup$

$\{[0_1,r_1,(b_r+4t+4)_1] \mid r=1,2,\ldots,4t+4$ and the $b_r$ are from an $(A,4t+4)$-system$\}$.

This is a set of base blocks for a bicyclic $DTS(v)$ under the permutation $\pi$. ∎

We now have:

**Theorem 2.1** *A bicyclic $DTS(v)$ admitting an automorphism consisting of two cycles of the same length, exists if and only if $v \equiv 4 \pmod 6$.*

**Proof.** If $v \equiv 4 \pmod{12}$, then there is a cyclic $DTS(v)$ admitting an automorphism $\alpha$ which is a cycle of length $v$. By considering $\alpha^2$ we see that this $DTS(v)$ is also bicyclic. If $v \equiv 10 \pmod{12}$, then Lemmas 2.1 through 2.5 give a bicyclic $DTS(v)$. This shows that the necessary condition of Lemma 2.1 is also sufficient. ∎

## 3. Automorphism Consists of Two Cycles of Different Lengths

In this section, we will consider bicyclic $DTS(v)$s with vertex set $\mathbf{Z}_N \times \{0\} \bigcup \mathbf{Z}_M \times \{1\}$ where $N < M$ and $N + M = v$ and with the automorphism $\pi = (0_0, 1_0, \ldots, (N-1)_0)(0_1, 1_1, \ldots, (M-1)_1)$. We need the following preliminary lemma.

**Lemma 3.1** *The fixed points of an autmorphism of a $DTS(v)$ form a subsystem.*

**Proof.** Suppose $x$ and $y$ are fixed under the automorphism $\pi$. The ordered pair $(x, y)$ occurs in exactly one block of the $DTS(v)$, say $b$. So $b$ is fixed under $\pi$ and the fixed points form a subsystem. ∎

We have the necessary conditions:

**Lemma 3.2** *If a bicyclic $DTS(v)$ admitting an automorphism $\pi$ consisting of a cycle of length $N$ and a cycle of length $M$ where $N < M$ exists, then $N \equiv 1$, 4, or 7 $\pmod{12}$ and $M = kN$ where $k \equiv 2 \pmod 3$.*

**Proof.** Suppose there is a bicyclic $DTS(v)$ with the vertex set and automorphism as described above. The points of $\mathbf{Z}_N \times \{0\}$ are fixed under $\pi^N$. Therefore, by Lemma 3.1, they form a subsystem. When $\pi$ is restricted to this subsystem, we see that the subsystem is cyclic. Therefore $N \equiv 1$, 4, or 7 (mod 12). So there must be some blocks of the bicyclic $DTS(v)$ with one vertex from $\mathbf{Z}_N \times \{0\}$ and two vertices from $\mathbf{Z}_M \times \{1\}$. As with the argument of Lemma 3.1, such blocks are fixed under $\pi^M$. So $N$ divides $M$.

There are five possible types of blocks in the design:

1. $[x_0, y_0, z_0]$ where $x_0, y_0, z_0 \in \mathbf{Z}_N \times \{0\}$,
2. $[x_1, y_1, z_1]$ where $x_1, y_1, z_1 \in \mathbf{Z}_M \times \{1\}$,

3. $[x_0, y_1, z_1]$ where $x_0 \in \mathbf{Z}_N \times \{0\}$ and $y_1, z_1 \in \mathbf{Z}_M \times \{1\}$,

4. $[y_1, z_1, x_0]$ where $x_0 \in \mathbf{Z}_N \times \{0\}$ and $y_1, z_1 \in \mathbf{Z}_M \times \{1\}$,

5. $[y_1, x_0, z_1]$ where $x_0 \in \mathbf{Z}_N \times \{0\}$ and $y_1, z_1 \in \mathbf{Z}_M \times \{1\}$.

The number of blocks in a $DTS(v)$ is $v(v-1)/3$, so the number of blocks of the first type is $N(N-1)/3$. The length of the orbit of a block of any of the other four types is $M$. So $M \mid (v(v-1)/3 - N(N-1)/3)$. That is, $M \mid M(M+2N-1)/3$ and therefore $M + 2N \equiv 1 \pmod 3$. This means that if $M = kN$, then $k \equiv 2 \pmod 3$. ∎

The following five lemmas show that the necessary conditions of Lemma 3.2 are sufficient.

**Lemma 3.3** *A bicyclic $DTS(v)$ admitting an automorphism $\pi$ consisting of a cycle of length $N$ and a cycle of length $M$ where $N < M$ exists if $N \equiv 1 \pmod{12}$, $N > 1$, and $M = kN$ where $k \equiv 2$ or $5 \pmod{12}$.*

**Proof.** Notice that for $N = 1$ the $DTS(v)$ is in fact rotational and these are described above. Now suppose $N = 12t + 1$ and $k = 3n + 2$ where $n \equiv 0$ or $1 \pmod 4$. Consider the set:

$\{[0_0, (3t-1+r)_1, (3t-r)_1], [0_0, (9t+r)_1, (9t-r)_1], [(9t-r)_1, (M-3t-1+r)_1, 0_0], [(3t-r)_1, (M-9t-2+r)_1, 0_0] \mid r = 1, 2, \ldots, 3t\} \bigcup$

$\{[(9t)_1, 0_0, (M-3t-1)_1]\} \bigcup$

$\{[0_1, r_1, (b_r + 12tn + 4t + n)_1] \mid r = 1, 2, \ldots, 12tn + 4t + n$ and the $b_r$ are from an $(A, 12tn + 4t + n)$-system$\}$.

This set of blocks when unioned with a set of base blocks for a cyclic $DTS(N)$ on $\mathbf{Z}_N \times \{0\}$ form a set of base blocks for a bicyclic $DTS(v)$. ∎

**Lemma 3.4** *A bicyclic $DTS(v)$ admitting an automorphism $\pi$ consisting of a cycle of length $N$ and a cycle of length $M$ where $N < M$ exists if $N \equiv 1 \pmod{12}$, $N > 1$, and $M = kN$ where $k \equiv 8$ or $11 \pmod{12}$.*

**Proof.** Again, if $N = 1$ the $DTS(v)$ is in fact rotational. Now suppose $N = 12t + 1$ and $k = 3n + 2$ where $n \equiv 2$ or $3 \pmod 4$. Consider the set:

$\{[0_0, (3t-1+r)_1, (3t-r)_1], [0_0, (9t+r)_1, (9t-r)_1], [(9t+1-r)_1, (M-3t+r)_1, 0_0], [(3t+1-r)_1, (M-9t-1+r)_1, 0_0] \mid r = 1, 2, \ldots 3t\} \bigcup$

$\{[(9t+1)_1, 0_0, (M-3t-1)_1]\} \bigcup$

$\{[0_1, r_1, (b_r + 12tn + 4t + n)_1] \mid r = 1, 2, \ldots, , 12tn + 4t + n$ and the $b_r$ are from a $(B, 12tn + 4t + n)$-system$\}$.

This set of blocks when unioned with a set of base blocks for a cyclic $DTS(N)$ on $\mathbf{Z}_N \times \{0\}$ form a set of base blocks for a bicyclic $DTS(v)$. ∎

**Lemma 3.5** *A bicyclic $DTS(v)$ admitting an automorphism $\pi$ consisting of a cycle of length $N$ and a cycle of length $M$ where $N < M$ exists if $N \equiv 4$ (mod 12), and $M = kN$ where $k \equiv 2$ (mod 3).*

**Proof.** Suppose $N = 12t + 4$. Consider the set:

$\{[0_0, (3t + r)_1, (3t + 1 - r)_1], [(9t + 3 - r)_1, (M - 3t - 1 + r)_1, 0_0] \mid r = 1, 2, \ldots, 3t + 1\} \bigcup$

$\{[0_0, (9t + 3 + r)_1, (9t + 3 - r)_1], [(3t + 2 - r)_1, (M - 9t - 3 + r)_1, 0_0] \mid r = 1, 2, \ldots, 3t\} \bigcup$

$\{[1_1, 0_0, (M - 6t - 2)_1], [(9t + 3)_1, 0_0, (M - 3t - 1)_1]\} \bigcup$

$\{[0_1, r_1, (b_r + 4kt - 4t + (4k - 5)/3)_1] \mid r = 1, 2, \ldots, 4kt - 4t + (4k - 5)/3$ and the $b_r$ are from an $(A, 4kt - 4t + (4k - 5)/3)$-system$\}$.

This set of blocks when unioned with a set of base blocks for a cyclic $DTS(N)$ on $\mathbf{Z}_N \times \{0\}$ form a set of base blocks for a bicyclic $DTS(v)$. ∎

**Lemma 3.6** *A bicyclic $DTS(v)$ admitting an automorphism $\pi$ consisting of a cycle of length $N$ and a cycle of length $M$ where $N < M$ exists if $N \equiv 7$ (mod 12), and $M = kN$ where $k \equiv 2$ or 11 (mod 12).*

**Proof.** Suppose $N = 12t + 7$ and $k = 3n + 2$ where $n \equiv 0$ or 3 (mod 4). Consider the set:

$\{[0_0, (3t + 1 + r)_1, (3t + 2 - r)_1], [(3t + 3 - r)_1, (M - 9t - 5 + r)_1, 0_0] \mid r = 1, 2, \ldots, 3t + 2\} \bigcup$

$\{[0_0, (9t + 5 + r)_1, (9t + 5 - r)_1], [(9t + 6 - r)_1, (M - 3t - 1 + r)_1, 0_0] \mid r = 1, 2, \ldots, 3t + 1\} \bigcup$

$\{[(9t + 6)_1, 0_0, (M - 3t - 2)_1]\} \bigcup$

$\{[0_1, r_1, (b_r + 12tn + 7n + 4t + 2)_1] \mid r = 1, 2, \ldots, 12tn + 7n + 4t + 2$ and the $b_r$ are from a $(B, 12tn + 7n + 4t + 2)$-system$\}$.

This set of blocks when unioned with a set of base blocks for a cyclic $DTS(N)$ on $\mathbf{Z}_N \times \{0\}$ form a set of base blocks for a bicyclic $DTS(v)$. ∎

**Lemma 3.7** *A bicyclic $DTS(v)$ admitting an automorphism $\pi$ consisting of a cycle of length $N$ and a cycle of length $M$ where $N < M$ exists if $N \equiv 7 \pmod{12}$, and $M = kN$ where $k \equiv 5$ or $8 \pmod{12}$.*

**Proof.** Suppose $N = 12t + 7$ and $k = 3n + 2$ where $n \equiv 1$ or $2 \pmod 4$. Consider the set:

$\{[0_0, (3t+1+r)_1, (3t+2-r)_1], [(3t+2-r)_1, (M-9t-6+r)_1, 0_0] \mid r = 1, 2, \ldots, 3t+2\} \bigcup$

$\{[0_0, (9t+5+r)_1, (9t+5-r)_1], [(9t+5-r)_1, (M-3t-2+r)_1, 0_0] \mid r = 1, 2, \ldots, 3t+1\} \bigcup$

$\{[(9t+5)_1, 0_0, (M-3t-2)_1]\} \bigcup$

$\{[0_1, r_1, (b_r + 12tn + 7n + 4t + 2)_1] \mid r = 1, 2, \ldots, 12tn + 7n + 4t + 2$ and the $b_r$ are from an $(A, 12tn + 7n + 4t + 2)$-system$\}$.

This set of blocks when unioned with a set of base blocks for a cyclic $DTS(N)$ on $\mathbf{Z}_N \times \{0\}$ form a set of base blocks for a bicyclic $DTS(v)$. ∎

Lemmas 3.2 through 3.7 combine to give:

**Theorem 3.1** *A bicyclic $DTS(v)$ admitting an automorphism $\pi$ consisting of a cycle of length $N$ and a cycle of length $M$ where $N < M$ exists if and only if $N \equiv 1$, 4, or $7 \pmod{12}$, and $M = kN$ where $k \equiv 2 \pmod 3$.*

Together, Theorems 2.1 and 3.1 give a complete classification of bicyclic directed triple systems.

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