Bicyclic Decompositions of K_v into Copies of $K_3 \bigcup \{e\}$

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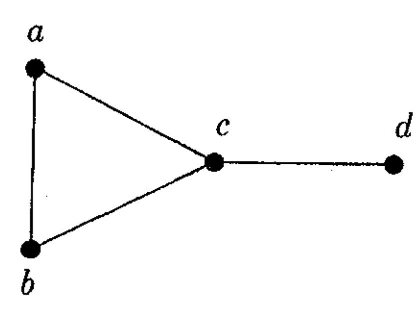
ABSTRACT. A decomposition of the complete graph on v vertices, K_v, into copies of K_3 with a pendant edge is called a "lollipop" system of order v, denoted $LS(v)$. We give necessary and sufficient conditions for the existence of a $LS(v)$ admitting an automorphism consisting of two disjoint cycles. We also give a brief proof that the previously known sufficient conditions for the existence of a cyclic $LS(v)$ are in fact necessary.

1 Introduction

A *G–design on H* is a set $\{g_1, g_2, \ldots, g_n\}$ of subgraphs of H (called *blocks*) such that $g_i \cong G$ for $i \in \{1, 2, \ldots, n\}$, $E(g_i) \bigcap E(g_j) = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{n} E(g_i) = E(H)$. Notice that a G–design on H is equivalent to a *G–decomposition* of H. Several G–designs on the complete graph, K_v, have been explored. In particular, necessary and sufficient conditions are known for such designs for $G \in \{K_3, S_n\} \bigcup \{ C_n \mid n \leq 50 \}$ where S_n denotes a star on $n+1$ vertices and C_n denotes a cycle on n vertices (see, for example, [1, 5, 10]). We are particularly interested in L–designs of K_v when L is the following graph:

$L =$ 

We denote L as given here by either $(a, b, c) - d$ or $(b, a, c) - d$. Bermond and Schönheim proved that an L–design on K_v exists if and only if $v \equiv 0$ or 1

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(mod 8) [2]. More generally, Hoffman and Kirkpatrick recently proved an L-design on λK_v exists if and only if $\lambda v(v-1) \equiv 0 \pmod 8$ [7]. Since the graph L is colloquially known as a "lollipop" [8], we refer to an L-design on K_v as a *lollipop system* of order v, denoted $LS(v)$.

An *automorphism* of a G-design on H is a permutation of $V(H)$ which fixes the set $\{g_1, g_2, \ldots, g_n\}$. Such an automorphism is said to be *cyclic* if it consists of a cycle of length $|V(H)|$ and is said to be *bicyclic* if it consists of a cycle of length M and a cycle of length N where $M + N = |V(H)|$.

A K_3-design of K_v is also known as a *Steiner triple system* of order v, denoted $STS(v)$. A cyclic $STS(v)$ exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$ [9] and necessary and sufficient conditions for the existence of a bicyclic $STS(v)$ are given in [3].

Bermond and Schönheim took advantage of "difference methods" in showing the existence of $LS(v)$s and proved that a cyclic $LS(v)$ exists if $v \equiv 1 \pmod 8$. We shall briefly show these conditions are, in fact, necessary. With the help of this result, we will then give necessary and sufficient conditions for the existence of a bicyclic $LS(v)$.

2 Cyclic Designs

For cyclic $LS(v)$s, we assume that the vertex set of K_v is $\{0, 1, \ldots, v-1\}$ and that the cyclic automorphism is $\pi_c = (0, 1, \ldots, v-1)$.

Theorem 2.1 *A cyclic $LS(v)$ exists if and only if $v \equiv 1 \pmod 8$.*

Proof. Suppose there is a cyclic $LS(v)$ where $v \equiv 0 \pmod 8$. There must be some g_i in such a design which contains the edge $(0, v/2)$. Applying $\pi_c^{v/2}$ to g_i, we see that $(0, v/2)$ is an edge of $\pi_c^{v/2}(g_i)$ and therefore $\pi_c^{v/2}(g_i) = g_i$. However, this is impossible. Therefore $v \not\equiv 0 \pmod 8$ and this, combined with the necessary condition for existence of a $LS(v)$, gives the necessary condition. Sufficiency is given in [2]. ∎

3 Bicyclic Designs

Throughout this section, we assume the vertex set of K_v is $\{0_0, 1_0, \ldots, (M-1)_0, 0_1, 1_1, \ldots, (N-1)_1\}$ where $M + N = v$ and we will construct $LS(v)$s admitting $\pi = (0_0, 1_0, \ldots, (N-1)_0)(0_1, 1_1, \ldots, (M-1)_1)$ as an automorphism. First we give necessary conditions for such a design.

Lemma 3.1 *A bicyclic $LS(v)$ admitting an automorphism consisting of a cycle of length M and a cycle of length N where $M = N = v/2$ does not exist.*

Proof. Suppose there is such a system. There must be some g_i in such a design which contains the edge $(0_0, (v/4)_0)$ As in Theorem 2.1, $\pi^{v/4}(g_i) = g_i$. Therefore edge $(0_0, (v/4)_0)$ must be in a copy of L of the form $(0_0, (v/4)_0, c) - d$ for some vertices c and d. But then we need $\pi^{v/4}(c) = c$ and $\pi^{v/4}(d) = d$ and this is a contradiction since no vertices are fixed under $\pi^{v/4}$. ∎

A *subdesign* of a G-design on K_v, $\{g_1, g_2, \ldots, g_n\}$, is a subset $\{g'_1, g'_2, \ldots, g'_M\} \subset \{g_1, g_2, \ldots, g_n\}$ which is a G-design on some complete subgraph of K_v.

Lemma 3.2 *If a bicyclic $LS(v)$ exists which admits an automorphism consisting of a cycle of length M and a cycle of length N where $M < N$, and the design does not contain a cyclic subsystem of order M on the vertices $\{0_0, 1_0, \ldots, (M-1)_0\}$, then $v \equiv 9 \pmod{24}$ and $N = 2M$.*

Proof. The automorphism π^M contains M fixed points. Suppose g_i is a block of such a design. We say an edge (x, y) is *absolutely fixed* by π if $\pi(x) = x$ and $\pi(y) = y$. Clearly if two or three edges of g_i are absolutely fixed under an automorphism, then all edges of g_i are absolutely fixed, and therefore all vertices of g_i are fixed under the automorphism. If exactly one edge of g_i is absolutely fixed under an automorphism, then the other three edges of g_i must be interchanged. This is only possible when $g_i = (a, b, c) - d$ where c and d are fixed and a and b are interchanged under the automorphism. Therefore, if some g_i of a bicyclic $LS(v)$ has exactly one absolutely fixed edge under π^M (such a g_i exists under the hypothesis that the system does not contain a subsystem on the fixed points), it must be that π^M consists of M fixed points and $N/2$ transpositions. Therefore $N = 2M$ and $v \equiv 0 \pmod 3$, which implies $v \equiv 0$ or $9 \pmod{24}$.

Now if $v \equiv 0 \pmod{24}$ then $M \equiv 0 \pmod 8$ and some g_i in such a design contains the edge $(0_0, (M/2)_0)$. We see that g_i must be fixed under $\pi^{M/2}$, a contradiction. ∎

Lemma 3.3 *If $v \equiv 9 \pmod{24}$, then there exists a bicyclic $LS(v)$ admitting an automorphism consisting of a cycle of length M and a cycle of length N where $N = 2M$ and $M + N = v$.*

Proof. Let $v = 24k + 9$, and so $M = 8k + 3$ and $N = 16k + 6$. We consider two cases based on the parity of k.

case 1. Suppose k is odd. Consider the set of blocks:

$\{(4k+1)_1, (12k+4)_1, 0_0) - (4k+1)_0\}$
$\bigcup \{(0_0, (\frac{3k+3}{2})_0, (\frac{3k+1}{2})_0) - (4k+2)_0\}$
$\bigcup \{(0_0, (\frac{3k+5}{2}+i)_0, (\frac{3k-1}{2}-i)_0) - (\frac{9k+3}{2})_0$ for $i = 0, 1, \ldots, \frac{k-3}{2}\}$
$\bigcup \{(0_0, (\frac{5k+5}{2}+i)_0, (\frac{5k+1}{2}-i)_0) - (6k+2)_0$ for $i = 0, 1, \ldots, \frac{k-3}{2}\}$
$\bigcup\{(0_0, (4k-i)_1, (4k+2+i)_1) - (4k+3+3i)_1$ for $i = 0, 1, \ldots, 4k\}$

case 2. Suppose k is even. Consider the set of blocks:

$\{(4k+1)_1, (12k+4)_1, 0_0) - (4k+1)_0\}$
$\bigcup \{(0_0, (\frac{3k+2}{2})_0, (\frac{3k}{2})_0) - (4k+1)_0\}$
$\bigcup \{(0_0, (\frac{3k+4}{2}+i)_0, (\frac{3k-2}{2}-i)_0) - (\frac{9k+2}{2})_0$ for $i = 0, 1, \ldots, \frac{k}{2}-2\}$
$\bigcup \{(0_0, (\frac{5k+4}{2}+i)_0, (\frac{5k}{2}-i)_0) - (6k+1)_0$ for $i = 0, 1, \ldots, \frac{k}{2}-1\}$
$\bigcup\{(0_0, (4k-i)_1, (4k+2+i)_1) - (4k+3+3i)_1$ for $i = 0, 1, \ldots, 4k\}$

In both cases, the set of blocks, along with the images of these blocks under the powers of π, form the desired design. ∎

Notice that under π^M, the design given in Lemma 3.3 has M fixed points, yet there is not a subdesign on these fixed points. This is contrary to the behavior of several previously studied graph and digraph decompositions (such as Steiner triple systems [6], directed triple systems [11], and Mendelsohn triple systems [4]).

Lemma 3.4 *If a bicyclic $LS(v)$ exists which admits an automorphism π consisting of a cycle of length M and a cycle of length N where $M < N$ and when π is restricted to $\{0_0, 1_0, \ldots, (M-1)_0\}$ we have a cyclic subsystem of order M on these points, then $M \equiv 1 \pmod 8$ and $N = kM$ where $k \equiv 7 \pmod 8$.*

Proof. Since there is a cyclic subsystem of order M, $M \equiv 1 \pmod 8$ is necessary by Theorem 2.1. In such a design, there must be a block of one of the following forms: $(a_1, b_1, c_1) - d_0$, $(a_1, b_1, c_0) - d_1$, or $(a_0, b_1, c_1) - d_1$. The points of $\{0_1, 1_1, \ldots, (N-1)_1\}$ are fixed under π^N and so the images of these blocks are respectively $(a_1, b_1, c_1) - \pi^N(d_0)$, $(a_1, b_1, \pi^N(c_0)) - d_1$, and $(\pi^N(a_0), b_1, c_1) - d_1$. In each case, π^N must fix vertices of $\{0_0, 1_0, \ldots, (M-1)_0\}$ and so $M \mid N$. If N is an even multiple of M, then the edge $(0_1, (N/2)_1)$ must be in some block of the design, and again as in Theorem 2.1, we get a contradiction. Therefore N must be an odd multiple of M. This condition, along with the fact that $v = M + N \equiv 0$ or $1 \pmod 8$, implies that $N = kM$ where $k \equiv 7 \pmod 8$. ∎

We now show the necessary conditions of Lemmas 3.1, 3.2 and 3.4 are in fact sufficient.

Theorem 3.1 *A bicyclic $LS(v)$ admitting an automorphism consisting of a cycle of length M and a cycle of length N, where $M \leq N$, exists if and only if*
(i) $N = 2M$ and $v = M + N \equiv 9 \pmod{24}$, or
(ii) $M \equiv 1 \pmod 8$ and $N = kM$ where $k \equiv 7 \pmod 8$.

Proof. Sufficiency for (i) is given in Lemma 3.3. Therefore, we need only show sufficiency in (ii). We do so in two cases.

case 1. Suppose $M \equiv 1 \pmod 8$ and $k \equiv 7 \pmod{16}$. Consider the set of blocks:

$$\left\{\left(0_0, \left(\tfrac{M-5}{4} - i\right)_1, \left(\tfrac{M(2k-1)-1}{4} + i\right)_1\right) - \left(\tfrac{M(4k-5)+5}{4} + 2i\right)_1 \text{ for } i = 0, 1, \ldots, \tfrac{M-5}{4}\right\}$$

$$\cup\left\{\left(0_0, \left(\tfrac{3M-7}{4} - i\right)_1, \left(\tfrac{M(2k+1)+1}{4} + i\right)_1\right) - \left(\tfrac{M(4k-2)+6}{4} + 2i\right)_1 \text{ for } i = 0, 1, \ldots, \tfrac{M-5}{4}\right\}$$

$$\cup\left\{\left(0_1, \left(\tfrac{3M(k-2)+9}{16}\right)_1, \left(\tfrac{3M(k-2)+25}{16}\right)_1\right) - \left(\tfrac{3M(k-6)+37}{16}\right)_0\right\}$$

$$\cup\left\{\left(0_1, \left(\tfrac{5M(k-2)+15}{16}\right)_1, \left(\tfrac{5M(k-2)+47}{16}\right)_1\right) - \left(\tfrac{5M(k-2)+39}{8}\right)_1\right\}$$

$$\cup\left\{\left(0_1, \left(\tfrac{3M(k-2)-7}{16} - i\right)_1, \left(\tfrac{3M(k-2)+41}{16} + i\right)_1\right) - \left(\tfrac{9M(k-2)+91}{16} + 2i\right)_1 \text{ for } i = 0, 1, \ldots, \tfrac{M(k-2)-29}{16}\right\}$$

$$\cup\left\{\left(0_1, \left(\tfrac{5M(k-2)-1}{16} - i\right)_1, \left(\tfrac{5M(k-2)+63}{16} + i\right)_1\right) - \left(\tfrac{3M(k-2)+25}{4} + 2i\right)_1 \text{ for } i = 0, 1, \ldots, \tfrac{M(k-2)-29}{16}\right\}.$$

case 2. Suppose $M \equiv 1 \pmod 8$ and $k \equiv 15 \pmod{16}$. Consider the set of blocks:

$$\left\{\left(0_0, \left(\tfrac{M-5}{4} - i\right)_1, \left(\tfrac{M(2k-1)-1}{4} + i\right)_1\right) - \left(\tfrac{M(4k-5)+5}{4} + 2i\right)_1 \text{ for } i = 0, 1, \ldots, \tfrac{M-5}{4}\right\}$$

$$\cup\left\{\left(0_0, \left(\tfrac{3M-7}{4} - i\right)_1, \left(\tfrac{M(2k+1)+1}{4} + i\right)_1\right) - \left(\tfrac{M(2k-1)+3}{2} + 2i\right)_1\right.$$

for $i = 0, 1, \ldots, \frac{M-5}{4}\}$

$$\cup\left\{\left(0_1, \left(\tfrac{3M(k-2)+17}{16}\right)_1, \left(\tfrac{3M(k-2)+33}{16}\right)_1\right) - \left(\tfrac{3M(k-6)+45}{16}\right)_0\right\}$$

$$\cup\left\{\left(0_1, \left(\tfrac{5M(k-2)+23}{16}\right)_1, \left(\tfrac{5M(k-2)+55}{16}\right)_1\right) - \left(\tfrac{5M(k-2)+47}{8}\right)_1\right\}$$

$$\cup\left\{\left(0_1, \left(\tfrac{3M(k-2)+1}{16} - i\right)_1, \left(\tfrac{3M(k-2)+49}{16} + i\right)_1\right) - \left(\tfrac{9M(k-2)+99}{16} + 2i\right)_1 \text{ for } i = 0, 1, \ldots, \tfrac{M(k-2)-21}{16}\right\}$$

$$\cup\left\{\left(0_1, \left(\tfrac{5M(k-2)+7}{16} - i\right)_1, \left(\tfrac{5M(k-2)+71}{16} + i\right)_1\right) - \left(\tfrac{12M(k-2)+116}{16} + 2i\right)_1 \text{ for } i = 0, 1, \ldots, \tfrac{M(k-2)-37}{16}\right\}.$$

In both cases, the set of blocks, along with the images of these blocks under the powers of π and a set of blocks for a cyclic $LS(M)$ on the point set $\{0_0, 1_0, \ldots, (M-1)_0\}$, form the desired design. ∎

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