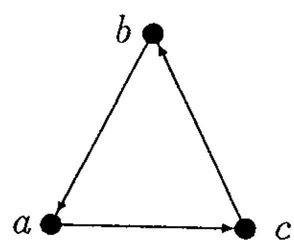
Triple Systems from Mixed Graphs

Robert Gardner
Institute of Mathematical and Physical Sciences
East Tennessee State University
Box 70296
Johnson City, TN 37614-0296

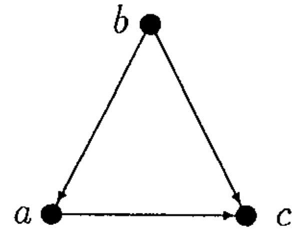
ABSTRACT. Necessary and sufficient conditions for the existence of some new triple systems are given by presenting decompositions of the complete mixed graph into certain mixed graphs on three vertices.

Introduction

et K_v (D_v) denote the complete graph (digraph) on v vertices. If g a graph (digraph) then a g*−decomposition* of K_v (D_v) is a set $\gamma =$ $g_1, g_2, \ldots, g_n\}$ of edge (arc) disjoint subgraphs of K_v (D_v) each of which isomorphic to g and such that $\bigcup_{i=1}^{n} E(g_i) = E(K_v)$ $\left(\bigcup_{i=1}^{n} A(g_i) = A(D_v)\right)$ 'here $E(G)$ ($A(G)$) is the edge (arc) set of G. Several triple systems are quivalent to either graph or digraph decompositions. A Steiner triple sys- em of order v is equivalent to a K_3−decomposition of K_v and it is well nown that such a system exists if and only if $v \equiv 1$ or $3 \pmod 6$. A Iendelsohn triple system of order v is equivalent to a decomposition of D_v ıto isomorphic copies of



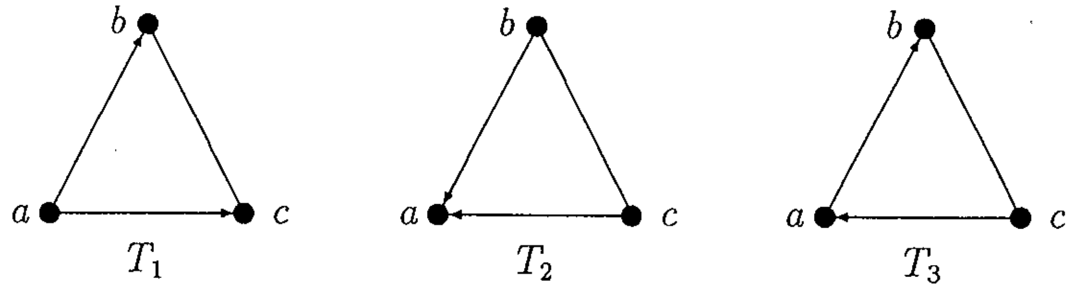
nd exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$ [3]. A directed triple ystem of order v is equivalent to a decomposition of D_v into isomorphic opies of



and exists if and only if $v \equiv 0$ or $1 \pmod 3$ [2]. For an additional discussion of triple systems and digraph decompositions, see [1].

A *mixed graph* on v vertices is an ordered pair (V, C) where V is a set of vertices, $|V| = v$, and C is a set of ordered and unordered pairs, denoted $[x, y]$ and (x, y) respectively, of elements of V. An ordered pair $[x, y] \in C$ is called an *arc* of (V, C) and an unordered pair $(x, y) \in C$ is called an *edge* of (V, C). The *complete mixed graph* on v vertices, denoted M_v, is the mixed graph (V, C) where, for every pair of distinct vertices $v_1, v_2 \in V$, we have $\{[v_1, v_2], [v_2, v_1], (v_1, v_2)\} \subset C$.

We are inspired by the above comments to study decompositions of M_v into copies of



We will denote T_i by the ordered triple $(a, b, c)_i$. We shall call such decompositions T_i*−triple systems* of order v, where $i = 1, 2$ or 3. In this paper, we give simple direct constructions of such systems.

2 The Constructions

We have the following necessary condition:

Lemma 2.1. *Let $i \in \{1, 2, 3\}$. If a T_i−triple system of order v exists, then $v \equiv 1$ (mod 2).*

Proof: In M_v, each vertex is in $v - 1$ edges and $2(v - 1)$ arcs. Each vertex of T_i is an element of either

1. one arc and one edge, or

2. two arcs.

So it is necessary that $2 \mid 3(v - 1)$. That is v, must be odd. □

We now show this necessary condition is sufficient for all but some small ılues of v. Throughout the remainder of this paper, we assume the vertex ɜt of M_v is Z_v.

'heorem 2.1. *A T_1–triple system of order v exists if and only if $v \equiv 1$ (mod 2).*

'roof: We consider two cases.
ase 1. Suppose $v \equiv 1 \pmod 4$, say $v = 4t + 1$. Consider the blocks:

$(j, 2t - i + j, 2t + 1 + i + j)_1$ for $i = 0, 1, \ldots, t-1$ and $j = 0, 1, \ldots, 4t$, and

$(j, 4t - i + j, 1 + i + j)_1$ for $i = 0, 1, \ldots, t-1$ and $j = 0, 1, \ldots, 4t$.

ase 2. Suppose $v \equiv 3 \pmod 4$, say $v = 4t + 3$. Consider the blocks:

$(j, 2t + 1 - i + j, 2t + 2 + i + j)_1$ for $i = 0, 1, \ldots, t$ and $j = 0, 1, \ldots, 4t$, and

$(j, 4t + 2 - i + j, 1 + i + j)_1$ for $i = 0, 1, \ldots, t-1$ and $j = 0, 1, \ldots, 4t$.

n both cases, the given blocks form a T_1–triple system. □

The *converse* of a mixed graph (V, C), is the mixed graph (V, C') where $C' = \{[v_2, v_1] \mid [v_1, v_2] \in C\} \bigcup \{(v_1, v_2) \mid (v_1, v_2) \in C\}$. Notice that the onverse of T_1 is T_2. Also, of course, the converse of M_v is M_v. Therefore he existence of a T_1–triple system implies the existence of a T_2–triple ystem and we have:

Theorem 2.2. *A T_2–triple system of order v exists if and only if $v \equiv 1$ (mod 2).*

Finally, we explore T_3–triple systems.

Theorem 2.3. *A T_3–triple system of order v exists if and only if $v \equiv 1$ (mod 2), $v \notin \{3, 5\}$.*

Proof: Clearly, such a design does not exist for $v = 3$. It is argued in the Appendix that such a design does not exist for $v = 5$. We now consider four cases.
Case 1. Suppose $v \equiv 1 \pmod 8$, say $v = 8t + 1$. Consider the blocks:

$(j, t + 2 + 2i + j, 1 + i + j)_3$ for $i = 0, 1, \ldots, 2t-1$ and $j = 0, 1, \ldots, 8t$,

$(j, 5t - 1 - 2i + j, 8t - i + j)_3$ for $i = 0, 1, \ldots, t-1$ and $j = 0, 1, \ldots, 8t$, and

$(j, t + 1 + 2i + j, 2t + 1 + i + j)_3$ for $i = 0, 1, \ldots, t-1$ and $j = 0, 1, \ldots, 8t$.

Case 2a. Suppose $v = 11$. Consider the blocks:

$(j, 1+j, 6+j)_3, (j, 3+j, 7+j)_3, (j, 2+j, 3+j)_3, (j, 6+j, 4+j)_3, (j, 9+j, 1+j)_3$ for $j = 0, 1, \ldots, 10$.

Case 2b. Suppose $v = 19$. Consider the blocks:

$(j, 1+j, 10+j)_3, (j, 2+j, 9+j)_3, (j, 3+j, 11+j)_3, (j, 4+j, 6+j)_3, (j, 5+j, 2+j)_3, (j, 6+j, 7+j)_3, (j, 7+j, 3+j)_3, (j, 11+j, 5+j)_3, (j, 15+j, 1+j)_3$ for $j = 0, 1, \ldots, 18$.

Case 2c. Suppose $v \equiv 3 \pmod 8$, $v \geq 27$, say $v = 8t + 3$ where $t \geq 3$. Consider the blocks:

$(j, 8t+2+j, 2t+j)_3$ and $(j, 6t+2+j, 3t+2+j)_3$, for $j = 0, 1, \ldots, 8t+2$,

$(j, 8t+1-i+j, 7t-2i+j)_3$ for $i = 0, 1, \ldots, t-1$ and for $j = 0, 1, \ldots, 8t+2$,

$(j, 7t+1-i+j, 5t-1-2i+j)_3$ for $i = 0, 1, \ldots, t-3$ and for $j = 0, 1, \ldots, 8t+2$,

$(j, 6t+1-i+j, 7t+1-2i+j)_3$ for $i = 0, 1, \ldots, t-1$ and for $j = 0, 1, \ldots, 8t+2$,

along with the following blocks if t is odd:

$(j, 5t+j, 8t+1+j)_3$ and $(j, 5t-1+j, 8t+2+j)_3$ for $j = 0, 1, \ldots, 8t+2$,

$(j, 5t-3+j, 8t-1+j)_3$ and $(j, t+j, 5t+1+j)_3$ for $j = 0, 1, \ldots, 8t+2$,

$(j, t+1+j, 5t+j)_3$ for $j = 0, 1, \ldots, 8t+2$ (omit if $t = 3$),

$(j, 3+2i+j, 3t+8+4i+j)_3$ for $i = 0, 1, \ldots, (t-5)/2$ and for $j = 0, 1, \ldots, 8t+2$,

$(j, 6+2i+j, 3t+10+4i+j)_3$ for $i = 0, 1, \ldots, (t-7)/2$ and for $j = 0, 1, \ldots, 8t+2$,

or the following blocks if t is even:

$(j, 5t+j, 8t+2+j)_3, (j, 5t-1+j, 8t+j)_3$, and $(j, 5t-3+j, 8t+1+j)_3$ for $j = 0, 1, \ldots, 8t+2$,

$(j, 5t-5+j, 8t-2+j)_3$ and $(j, t+j, 5t+1+j)_3$ for $j = 0, 1, \ldots, 8t+2$,

$(j, t+1+j, 5t+j)_3$ for $j = 0, 1, \ldots, 8t+2$ (omit if $t = 4$),

$(j, 4+2i+j, 3t+10+4i+j)_3$ for $i = 0, 1, \ldots, (t-6)/2$ and for $j = 0, 1, \ldots, 8t+2$,

$(j, 7+2i+j, 3t+12+4i+j)_3$ for $i = 0, 1, \dots, (t-8)/2$ and for $j = 0, 1, \dots, 8t+2$.

Case 3. Suppose $v \equiv 5 \pmod 8$, $v \geq 13$, say $v = 8t+5$ where $t \geq 1$. Consider the blocks:

$(j, 8t+4+j, 2t+j)_3$ and $(j, 6t+4+j, 3t+3+j)_3$ for $j = 0, 1, \dots, 8t+4$,

$(j, 8t+3-i+j, 7t+1-2i+j)_3$ for $i = 0, 1, \dots, t-2$ and for $j = 0, 1, \dots, 8t+4$,

$(j, 7t+4-i+j, 5t+2-2i+j)_3$ for $i = 0, 1, \dots, t-2$ and for $j = 0, 1, \dots, 8t+4$,

$(j, 6t+3-i+j, 7t+4-2i+j)_3$ for $i = 0, 1, \dots, t$ and for $j = 0, 1, \dots, 8t+4$,

along with the following blocks if t is odd:

$(j, 5t+1+j, 8t+4+j)_3$ and $(j, 5t+j, 8t+2+j)_3$ for $j = 0, 1, \dots, 8t+4$,

$(j, 5t-2-4i+j, 8t+3-2i+j)_3$ for $i = 0, 1, \dots, (t-3)/2$ and for $j = 0, 1, \dots, 8t+4$,

$(j, 3t+2+4i+j, 7t+3+2i+j)_3$ for $i = 0, 1, \dots, (t-3)/2$ and for $j = 0, 1, \dots, 8t+4$,

or the following blocks if t is even:

$(j, 5t+1+j, 8t+3+j)_3$ and $(j, 5t+j, 8t+4+j)_3$ for $j = 0, 1, \dots, 8t+4$,

$(j, t+2-2i+j, 5t+3-4i+j)_3$ for $i = 0, 1, \dots, (t-2)/2$ and for $j = 0, 1, \dots, 8t+4$,

$(j, t-1-2i+j, 5t+1-4i+j)_3$ for $i = 0, 1, \dots, (t-4)/2$ and for $j = 0, 1, \dots, 8t+4$.

Case 4. Suppose $v \equiv 7 \pmod 8$, say $v = 8t-1$. Consider the blocks:

$(j, 5t-2-2i+j, 8t-2-i+j)_3$ for $i = 0, 1, \dots, t-1$ and $j = 0, 1, \dots, 8t-2$,

$(j, 3t-2-2i+j, 3t-1-i+j)_3$ for $i = 0, 1, \dots, t-2$ and $j = 0, 1, \dots, 8t-2$, and

$(j, t+1+2i+j, 1+i+j)_3$ for $i = 0, 1, \dots, 2t-1$ and $j = 0, 1, \dots, 8t-2$.

□

Theorems 2.1, 2.2 and 2.3 give necessary and sufficient conditions for the existence of mixed triple systems.

References

[1] A. Hartman and E. Mendelsohn, The last of the triple systems, *Ars Combinatoria* **22** (1986), 25–41.

[2] S. Hung and N. Mendelsohn, Directed triple systems, *J. Combin. Theory Series A* **14** (1973), 310–318.

[3] N. Mendelsohn, A natural generalization of Steiner triple systems, *Computers in Number Theory*, eds. A. O. Atkin and B. Birch, Academic Press, London, 1971.

APPENDIX - A T_3-Triple System of Order 5 Does Not Exist

Suppose a T_3−triple system of order 5 exists. Then it consists of 10 copies of T_3. In a copy of T_3, $(a,b,c)_3$, we say that vertex a (which has in-degree 1 and out-degree 1) is of *type A*, vertex b (which has in-degree 1 and is incident to an edge of this T_3) is of *type B*, and vertex c (which has out-degree 1 and is incident to an edge of this T_3) is of *type C*. In the hypothesized design, let x be the number of times a given vertex appears in a copy of T_3 as a type A vertex, let y be the number of times a given vertex appears in a copy of T_3 as a type B vertex, and let z be the number of times a given vertex appears in a copy of T_3 as a type C vertex. In M_5, each vertex is of out-degree 4, so it is necessary that $x+y=4$. Similarly, $y+z=4$ and $x+z=4$. We therefore see that a given vertex in a T_3−triple system of order 5 must appear twice as a type A vertex, twice as a type B vertex, and twice as a type C vertex. Hence, the existence of the hypothesized system is equivalent to associating a vertex with each edge of K_5 in a copy of T_3 as follows:

$$(a_1,0,1)_3,(a_2,0,2),(a_3,0,3)_3,(a_4,0,4)_3,(a_5,1,2)_3,$$

$$(a_6,1,3)_3,(a_7,1,4)_3,(a_8,2,3)_3,(a_9,2,4)_3,(a_{10},3,4)_3,$$

where each a_i is distinct from the other two vertices in the copies of T_3 which contain it (therefore we immediately have $3^{10}=59,049$ possible cases). In addition, the multiset $\{a_1,a_2,\ldots,a_{10}\}$ must contain each element of $\{0,1,2,3,4\}$ twice (we call such a choice of the a_i's *viable*). Finally, the arcs contained in this collection of 10 T_3's must contain each arc of D_5. The author has performed a computer search of the 3^{10} possible choices of the a_i's and found that 906 of them are viable. However, none produce a T_3−triple system of order 5. Therefore, no such system exists.