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Note

Steiner Triple Systems with Near-Rotational Automorphisms

ROBERT B. GARDNER

Division of Mathematics, Department of Algebra, Combinatorics, and Analysis, Auburn University, Alabama 36849

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A Steiner triple system of order v, denoted STS(v), is said to be k-near-rotational if it admits an automorphism consisting of three fixed points and k cycles of length $(v-3)/k$. In this paper, we show that, for $n \geqslant 1$, a $2n$-near-rotational STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6), $v \equiv 3$ (mod $2n$), and $v \neq 13$ or 21 when $n = 1$. Also, for $n \geqslant 1$, a $3n$-near-rotational STS(v) exists if and only if $v \equiv 3$ (mod 6) and $v \equiv 3$ (mod $3n$).

1. INTRODUCTION

A *Steiner triple system* of order v, denoted STS(v), is a v-element set, X, of points, together with a set β, of unordered triples of elements of X, called *blocks*, such that any two points of X are together in exactly one block of β. It is well known that a STS(v) exists if and only in $v \equiv 1$ or 3 (mod 6). An *automorphism* of a STS(v) is a permutation, π, of X which fixes β. A permutation π of a v-element set is said to be of *type* $[\pi] = [p_1, p_2, ..., p_v]$ if the disjoint cyclic decomposition of π contains p_i cycles of length i. The *orbit* of a block under an automorphism, π, is the image of the block under the powers of π. A set of blocks, **B**, is said to be a *set of base blocks for a* STS(v) *under the permutation* π if the orbits of the blocks of **B** produce the STS(v) and exactly one block of **B** occurs in each orbit.

Several types of automorphisms have been explored in connection with the problem of determining the values of v for which there is a STS(v) admitting the automorphism. In particular, a STS(v) admitting an automorphism of type $[1, 0, ..., 0, k, 0, ..., 0]$ is called a *k-rotational* STS(v). The question of existence for k-rotational STS(v) has been solved for $k = 1, 2, 3, 4$, and 6 [3, 8]. It is fairly easy to see that the fixed points of

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an automorphism form a subsystem. Since a STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6), the number of fixed points is also 1 or 3 (mod 6). Therefore, a natural question to ask is "What is the spectrum of values of v for which there is a STS(v) admitting an automorphism consisting of three fixed points and k cycles each of length $(v-3)/k$?" We will call such a design a *k-near-rotational* STS(v).

2. The Existence of $2n$-Near-Rotational Steiner Triple Systems

A $2n$-near-rotational STS(v) admits an automorphism of the type $[3, 0, 0, ..., 0, 2n, 0, ..., 0]$. We will first show existence for 2-near-rotational STS(v) and then deal with the general $2n$-near-rotational case. We need the existence of certain other types of Steiner triple systems. A STS(v) admitting an automorphism consisting of a single cycle is called *cyclic* and such systems exist if and only if $v \equiv 1$ or 3 (mod 6) and $v \neq 9$ [5, 6, 7, and 9]. A STS(v) admitting an automorphism of type $[1, 1, 0, ..., 0, k, 0, ..., 0]$ is said to be *k-transrotational.* A 1-transrotational STS(v) exists if and only if $v \equiv 1, 7, 9$, or 15 (mod 24) [4]. A STS(v) admitting an automorphism of type $[0, 0, 1, 0, ..., 0, 1, 0, 0, 0]$ exists if and only if $v \equiv 3$ (mod 6) [2].

Theorem 2.3. *A 2-near-rotational* STS(v) *exists if and only if* $v \equiv 1$ *or* 3 (mod 6) *and* $v \neq 13$ *or* 21.

Proof. If $v \equiv 1, 7, 9$, or 15 (mod 24) then there exists a 1-transrotational STS(v) admitting the relevant automorphism π of type $[1, 1, 0, ..., 0, 1, 0, 0, 0]$. If we consider the same STS(v) under the automorphism π^2 then we see that it is also 2-near-rotational.

We now answer the question of existence in the remaining cases by considering blocks on the set $X = \mathbf{Z}_N \times \{1, 2\} \cup \{\infty_1, \infty_2, \infty_3\}$. We put the automorphism $\pi = (\infty_1)(\infty_2)(\infty_3)(0_1, 1_1, ..., (N-1)_1)(0_2, 1_2, ..., (N-1)_2)$ on this set, where $N = (v-3)/2$. With the pair (x_i, y_i) we associate the *pure difference of type i* defined as $\min\{(x-y)(\text{mod } N), (y-x)(\text{mod } N)\}$. With the pair (x_1, y_2) we associate the *mixed difference* $(y-x)(\text{mod } N)$. The construction of a 2-near-rotational STS(v) is equivalent to partitioning the collection of these differences into sets of differences associated with blocks which are base blocks under π.

For $v = 13$, the set of mixed differences, $\{0, 1, 2, 3, 4\}$, the set of pure differences of type 1, $\{1, 2\}$, and the set of pure differences of type 2, $\{1, 2\}$, must be partitioned into sets of differences associated with base blocks of a STS(13) under π. A mixed difference must be used in a base block of the form (∞_k, x_i, y_j), $i \neq j$, for $k = 1, 2, 3$. This leaves two mixed differences and a total of four pure differences. These clearly cannot be partitioned into sets of differences associated with base blocks. For $v=21$, a similar exhaustive search reveals that a 2-near-rotational STS(21) does not exist. For more information on the difference method of construction, see Anderson [1].

In the following cases, base blocks for a 2-near-rotational STS(v) under π are presented:

Case 1a. If $v=27$ then take the blocks

$$(\infty_1, \infty_2, \infty_3), (0_1, 2_1, 3_1), (0_1, 4_1, 8_1), (\infty_1, 0_1, 6_1), (0_2, 4_2, 8_2),$$
$$(\infty_1, 0_2, 6_2), (0_1, 2_2, 3_2), (0_1, 1_2, 4_2), (0_1, 0_2, 5_2), (0_1, 8_2, 10_2),$$
$$(11_2, 0_1, 5_1), (\infty_2, 0_1, 7_2), (\infty_3, 0_1, 9_2).$$

Case 1b. If $v \equiv 3 \pmod{24}$, say $v=24k+3$, where $k \geqslant 2$, then take the blocks

$$(\infty_1, \infty_2, \infty_3), (\infty_1, 0_1, (6k)_1), (\infty_1, 0_2, (6k)_2), (\infty_2, 0_1, (9k-1)_2),$$
$$(\infty_3, 0_1, (11k-1)_2), (0_1, (4k)_1, (8k)_1), (0_2, (4k)_2, (8k)_2),$$
$$(0_1, (6k)_2, (11k)_2),$$
$$(0_2, (3k-1-r)_2, (3k+r)_2) \quad \text{for } r=0, 1, ..., k-1,$$
$$(0_2, (5k-1-r)_2, (5k+1+r)_2) \quad \text{for } r=0, 1, ..., k-2,$$
$$(0_2, (9k-r)_1, (9k+1+r)_1) \quad \text{for } r=0, 1, ..., 3k-1,$$
$$(0_2, (3k-r)_1, (3k+2+r)_1) \quad \text{for } r=0, 1, ..., 2k-2,$$
$$(0_2, (5k+1+r)_1, (k-1-r)_1) \quad \text{for } r=0, 1, ..., k-2.$$

Case 2a. If $v=37$ then take the blocks

$$(\infty_1, \infty_2, \infty_3), (\infty_1, 0_1, 7_2), (\infty_2, 0_1, 11_2), (\infty_3, 0_1, 12_2),$$
$$(0_1, 4_1, 10_1), (0_2, 4_2, 10_2), (0_2, 5_2, 8_2),$$
$$(0_2, 13_1, 14_1), (0_2, 12_1, 15_1), (0_2, 11_1, 16_1),$$
$$(0_2, 0_1, 9_1), (0_2, 1_1, 3_1), (0_1, 9_2, 10_2), (0_1, 13_2, 15_2).$$

Case 2b. If $v \equiv 13 \pmod{24}$, say $v=24k+13$, where $k \geqslant 2$, then take the blocks

$(\infty_1, \infty_2, \infty_3)$, $(0_2, (2k)_2, (3k+1)_2)$,

$(0_2, (4k-3)_1, (2k-3)_1)$, $(\infty_1, 0_1, (9k+3)_2)$,

$(0_1, (5k+1-r)_1, (5k+2+r)_1)$ for $r=0, 1, ..., k$,

$(0_1, (3k-r)_1, (3k+2+r)_1)$ for $r=0, 1, ..., k-2$,

$(0_1, (3k-r)_2, (3k+1+r)_2)$ for $r=0, 1, ..., 3k$,

$(0_1, (9k+2-r)_2, (9k+4+r)_2)$ for $r=0, 1, ..., k-2, k, k+1, ..., 3k$.

One of the above blocks is of the form $(0_1, a_2, (a+3k+1)_2)$. Replace it with the block $(0_2, (12k-a)_1, (9k-1-a)_1)$.

Another block is of the form $(0_1, b_2, (b+k+1)_2)$. Omit it and add the blocks $(\infty_2, 0_1, b_2)$ and $(\infty_3, 0_1, (b+k+1)_2)$.

Case 3a. If $v=19$ then take the blocks

$(\infty_1, \infty_2, \infty_3)$, $(\infty_1, 0_1, 4_1)$, $(\infty_1, 0_2, 4_2)$, $(\infty_2, 0_1, 5_2)$, $(\infty_3, 0_1, 7_2)$

$(0_1, 1_1, 3_1)$, $(0_1, 1_2, 2_2)$, $(0_1, 0_2, 3_2)$, $(0_1, 4_2, 6_2)$

Case 3b. If $v \equiv 19 \pmod{24}$, say $v=24k+19$, where $k \geqslant 1$, then take the blocks

$(\infty_1, \infty_2, \infty_3)$, $(\infty_1, 0_1, (6k+4)_1)$, $(\infty_1, 0_2, (6k+4)_2)$,

$(\infty_2, 0_1, (8k+4)_2)$, $(\infty_3, 0_1, (10k+6)_2)$, $(0_2, (8k+6)_1, (10k+7)_1)$,

$(0_2, (7k+5)_1, (11k+8)_1)$, $(0_2, (2k+1)_2, (4k+3)_2)$, $(0_2, 1_1, (3k+3)_1)$

$(0_1, (3k+1-r)_1, (3k+3+r)_1)$ for $0, 1, ..., k-1$,

$(0_1, (5k+3-r)_2, (5k+4+r)_2)$ for $r=0, 1, ..., k-1$,

$(0_1, (3k+1-r)_2, (3k+2+r)_2)$ for $r=0, 1, ..., k-1, k+1, k+2, ..., 2k,$
$2k+2, 2k+3, ..., 3k+1$,

$(0_1, (9k+4-r)_2, (9k+6+r)_2)$ for $r=0, 1, ..., k-1, k+1, k+2, ..., 3k$.

Case 4. If $v \equiv 21 \pmod{24}$, say $v=24k+21$, where $k \geqslant 1$, then take the blocks

$(\infty_1, \infty_2, \infty_3)$, $(0_2, (4k+3)_2, (8k+6)_2)$, $(\infty_1, 0_1, k_2)$,

$(\infty_2, 0_1, (5k+3)_2)$, $(\infty_3, 0_1, (9k+6)_2)$,

$(0_1, (3k+1-r)_2, (3k+2+r)_2)$ for $r=0, 1, ..., 2k, 2k+2,$
$2k+3, ..., 3k+1$,

$(0_1, (9k+5-r)_2, (9k+7+r)_2)$ for $r=0, 1, ..., 3k+1$.

Plus, take the base blocks for a cyclic STS(N) on the set $\mathbf{Z}_N \times \{1\}$, where $N = (v-3)/2$. This can be done since $N \equiv 3 \pmod 6$ and $N \neq 9$. ▮

We now turn our attention to $2n$-near-rotational STS(v).

THEOREM 2.4. *A $2n$-near-rotational* STS(v) *exists if and only if* $v \equiv 1$ *or* 3 (mod 6), $v \equiv 3 \pmod{2n}$, *and* $v \neq 13$ *or* 21 *when* $n = 1$.

Proof. Since a Steiner triple system exists if and only if $v \equiv 1$ or 3 (mod 6), this is a trivial necessary condition. Also, a $2n$-near-rotational STS(v) has an automorphism consisting of three fixed points and $2n$ cycles, so it is necessary that $2n \mid (v-3)$.

In Theorem 2.3, we saw that a 2-near-rotational STS(v) does not exist for $v = 13$ or 21. However, a 6-near-rotational STS(21) does exist. By previously stated results, there exists a STS(21), admitting an automorphism π of type [0, 0, 1, 0, ..., 0, 1, 0, 0, 0]. This system is also 6-near-rotational as can be seen by considering π^6. In general, if we take any 2-near-rotational STS(v) admitting the relevant automorphism π, then by taking the automorphism π^n we see that the STS(v) is also $2n$-near-rotational. ▮

3. The Existence of $3n$-Near-Rotational Steiner Triple Systems

A $3n$-near-rotational STS(v) admits an automorphism of the type [3, 0, 0, ..., 0, $3n$, 0, ..., 0]. The construction of these trivially follows from a result of Calahan [2].

THEOREM 3.1. *A $3n$-near-rotational* STS(v) *exists if and only if* $v \equiv 3$ (mod 6) *and* $v \equiv 3 \pmod{3n}$,

Proof. The conditions are necessary, since $3n \mid (v-3)$. Sufficiency is established by applying an above mentioned result. If v satisfies the necessary conditions, then there is a STS(v) admitting an automorphism π of type [0, 0, 1, 0, ..., 0, 1, 0, 0, 0] (see [2]). The automorphism π^{3n} is then of type [3, 0, ..., 0, $3n$, 0, ..., 0] and the STS(v) is also $3n$-near-rotational. ▮

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