Decompositions of Uniform Complete Directed Multigraphs Into Each of the Orientations of a 4-Cycle

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Abstract. We present necessary and sufficient conditions for the decomposition of λ times the complete directed digraph, D_v^λ, into each of the orientations of a 4−cycle. In our constructions, we also give necessary and sufficient conditions for such decompositions which admit cyclic or rotational automorphisms.

1 Introduction

The area of graph (and directed graph) decompositions includes not only a large body of theoretical studies, but applications in such areas as coding theory, X-ray crystallography, radioastronomy, computer and communication networks, serology and genetics [2]. In particular, graph decompositions are intimately related to combinatorial designs. For example, a K_3 decomposition of K_v is equivalent to a Steiner triple system of order v [13]. A 3-circuit decomposition of the complete symmetric directed graph on v vertices is equivalent to a Mendelsohn triple system of order v [11]. The orientation of a 3-cycle that is *not* a 3-circuit is called a transitive triple. A transitive triple decomposition of the complete symmetric directed graph on v vertices is equivalent to a directed triple system of order v [10].

We denote the complete symmetric directed graph as D_v and the uniform complete directed multigraph of multiplicity λ as D_v^λ. For g a directed graph, a $g-$*decomposition* of D_v^λ is a multiset $\gamma = \{g_1, g_2, \ldots, g_n\}$ of isomorphic copies of g (called *blocks*) such that $\bigcup_{i=1}^{n} A(g_i) = A(D_v^\lambda)$ where $A(G)$ denotes the arc multiset of directed graph G (notice that we are using the union notation between multisets to denote a union operation with multiple copies counted multiply).

There are two orientations of a 3-cycle: the 3-circuit and the transitive triple. This leads us to study two types of triple systems. 3-circuit decompositions of D_v^λ are explored in [1]:

Theorem 1.1 *A* 3*-circuit decomposition of* D_v^λ *exists if and only if* $\lambda v(v-1) \equiv 0 \ (mod\, 3)$*, except for the case* $v = 6$ *and* $\lambda = 1$.

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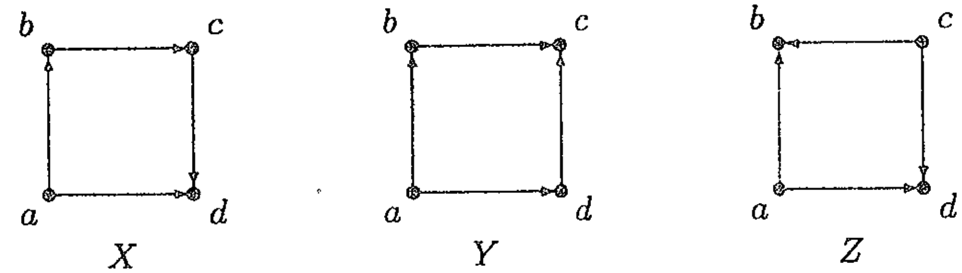
Transitive triple decompositions of D_v^λ are explored in [16]:

Theorem 1.2 *A transitive triple decomposition of D_v^λ exists if and only if $\lambda v(v-1) \equiv 0 \ (mod\ 3)$, except for the case $v = 2$.*

We concentrate on results analogous to Theorems 1.1 and 1.2 for orientations of 4-cycles. There are four orientations of a 4−cycle: the 4−circuit and the digraphs



We denote these digraphs as $[a,b,c,d]_X$, $[a,b,c,d]_Y$, and $[a,b,c,d]_Z$, respectively, and we denote the 4−circuit with arcs (a,b), (b,c), (c,d) and (d,a) as $[a,b,c,d]_C$. A 4−circuit decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 4$ [15]. An X−decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 5$, a Y−decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \notin \{4,5\}$, and a Z−decomposition of D_v exists if and only if $v \equiv 1$ (mod 4) [9].

An *automorphism* of a g−decomposition of D_v^λ is a permutation of the vertex set of D_v^λ which fixes the multiset γ. A collection of blocks $\{g_1, g_2, \ldots, g_m\}$ in a g−decomposition admitting an automorphism α is a collection of *base blocks* if $\gamma = \bigcup_{i=1}^{m} \{\alpha^j(g_i) \mid j \in \mathbb{Z}\}$ and the collection is minimal (that is, no copy of g is repeated an unnecessary number of times). The *orbit* of base block g_i is the set $\{\alpha^j(g_i) \mid j \in \mathbb{Z}\}$ and the *length* of an orbit is the cardinality of this set. An automorphism of a g−decomposition of D_v^λ is *cyclic* if it consists of a single cycle of length v. An automorphism is *rotational* if it consists of a fixed point and a cycle of length $v-1$. A number of graph (and directed graph) decompositions have been explored which admit cyclic or rotational automorphisms. In particular, cyclic and rotational automorphisms of Steiner triple systems are explored in [4, 13]. Cyclic and rotational automorphisms of transitive triple decompositions of D_v^λ are explored in [3, 5, 6, 8]. In this paper, we explore cyclic and rotational decompositions of D_v^λ into each of the orientations of a 4−cycle. These results will lead us to necessary and sufficient conditions for the existence of decompositions of D_v^λ.

2 Cyclic Decompositions

Throughout this section, we let the vertex set of D_v^λ be $\mathbb{Z}_v$ and suppose that a cyclic decomposition admits the automorphism $\alpha = (0, 1, 2\ldots, v-1)$. With arc (x, y) we associate the *difference* $y - x \pmod v$. Notice that the totality of differences associated with the arc multiset of D_v^λ is $\lambda \times \{1, 2, \ldots, v-1\}$.

In a collection of base blocks for a cyclic 4–circuit decomposition of D_v^λ, the differences associated with a copy of the 4–circuit must satisfy one of the following conditions:

A. There are four (not necessarily distinct) differences a_i, b_i, c_i, d_i such that $a_i + b_i + c_i + d_i \equiv 0 \pmod v$ and the length of the orbit of the base block is v.

B. v is even and there are two distinct differences a_i, b_i such that $a_i + b_i \equiv v/2 \pmod v$ and the length of the orbit of the base block is $v/2$.

C. $v \equiv 0 \pmod 4$ and there is the single difference $v/4$ or the single difference $3v/4$ and the length of the orbit of the base block is $v/4$.

We have the following necessary conditions.

Lemma 2.1 *If a cyclic 4–circuit decomposition of D_v^λ exists, then*

1. *if $v \equiv 0$ or $3 \pmod 4$ then $\lambda \equiv 0 \pmod 2$, or*

2. *$v \equiv 1 \pmod 4$, or*

3. *if $v \equiv 2 \pmod 4$ then $\lambda \equiv 0 \pmod 4$.*

Proof. First, we certainly need $\lambda v(v-1) \equiv 0 \pmod 4$.

Suppose there exists such a decomposition with $v \equiv 0 \pmod 4$ and $\lambda \equiv 1 \pmod 2$. Then it is necessary to partition the difference multiset $\lambda \times \{1, 2, \ldots, v-1\}$ into differences associated with base blocks in such a way as to satisfy conditions A, B, and C. The total number of differences is $\lambda(v-1)$ which is odd. Since under conditions A and B an even number of differences are associated with each base block, we must have an odd number of base blocks whose associated differences satisfy condition C. The sum of the differences associated with a base block satisfying condition A or B is a multiple of $v/2$ and the difference associated with a base block satisfying condition C is congruent to $v/4$ modulo $v/2$. Therefore the total sum of differences associated with base blocks for such a decomposition is congruent to $v/4$ modulo $v/2$. However, the total sum of differences is $\frac{\lambda v(v-1)}{2} \equiv 0$ (mod $v/2$). This contradiction shows that no such decomposition exists.

Suppose there exists such a decomposition with $v \equiv 2 \pmod 4$ and $\lambda \equiv 2 \pmod 4$. Then it is necessary to partition the difference multiset $\lambda \times \{1, 2, \ldots, v-1\}$ into differences associated with base blocks in such a way as to satisfy conditions A and B. The total number of differences is $\lambda(v-1) \equiv 2 \pmod 4$. Therefore we must have an odd number of base blocks whose associated differences satisfy condition B. Under condition B, we see that each relevant block has associated with it one even difference and one odd difference. In this way, an odd number of even differences are used in base blocks whose associated differences satisfy condition B. The remainder of base blocks have associated differences which satisfy condition A, and therefore each such block has an even number of even differences associated with it. We therefore must have an odd number of even differences. However, since λ is even, the total number of even differences is even. This contradiction shows that no such decomposition exists. ∎

We now show that these necessary conditions are in fact sufficient.

Theorem 2.1 *A cyclic 4−circuit decomposition of D_v^λ exists if and only if*

1. $v \equiv 0$ *or* $3 \pmod 4$ *and* $\lambda \equiv 0 \pmod 2$, *or*
2. $v \equiv 1 \pmod 4$, *or*
3. $v \equiv 2 \pmod 4$ *and* $\lambda \equiv 0 \pmod 4$.

Proof. The necessary conditions follow from Lemma 2.1. We show sufficiency in five cases.

Case 1. Suppose $v \equiv 0 \pmod 8$ and $\lambda \equiv 0 \pmod 2$, say $v = 8t$. Consider the blocks:

$[0, 2t, 4t, 6t]_C$, $[0, 6t, 4t, 2t]_C$, $[0, 4t-1, 8t-1, 4t]_C$, $[0, 2t-1, 4t, 2t+1]_C$
$[0, 1+2i, 3+4i, 2+2i]_C$ for $i = 0, 1, 2, \ldots, 2t-2$
$[0, 1+2i, 3+4i, 2+2i]_C$ for $i = 0, 1, 2, \ldots, t-2$, and
$[0, 4t-2-2i, 8t-3-4i, 4t-1-2i]_C$ for $i = 0, 1, 2, \ldots, t-2$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. For general λ even. we take $\lambda/2$ copies of these blocks.

Case 2. Suppose $v \equiv 4 \pmod 8$ and $\lambda \equiv 0 \pmod 2$, say $v = 8t + 4$. Consider the blocks:

$[0, 2t+1, 4t+2, 6t+3]_C, [0, 6t+3, 4t+2, 2t+1]_C, [0, 4t+1, 8t+3, 4t+2]_C$,
$[0, 1+2i, 3+4i, 2+2i]_C$ for $i = 0, 1, 2, \ldots, 2t-1$,
$[0, 1+2i, 3+4i, 2+2i]_C$ for $i = 0, 1, 2, \ldots, t-1$, and
$[0, 4t-2i, 8t+1-4i, 4t+1-2i]_C$ for $i = 0, 1, 2, \ldots, t-1$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. For general λ even, we take $\lambda/2$ copies of these blocks.

Case 3. Suppose $v \equiv 1 \pmod 4$. Micale and Pennisi [12] have shown that a cyclic 4–circuit decomposition of D_v exists. For general λ even, we take λ copies of the blocks of such a decomposition.

Case 4. Suppose $v \equiv 2 \pmod 4$ and $\lambda \equiv 0 \pmod 4$, say $v = 4t + 2$. Consider the blocks:

$[0, 2t, 4t+1, 2t+1]_C, [0, 2t-1, 4t, 2t+1]_C$,
$3 \times [0, 1+2i, 4t+1, 4t-2i]_C$ for $i = 0, 1, 2, \ldots, t-1$, and
$[0, 1+2i, 4t+1, 4t-2i]_C$ for $i = 0, 1, 2, \ldots, t-2$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 4$. For general $\lambda \equiv 0 \pmod 4$, we take $\lambda/4$ copies of these blocks.

Case 5. Suppose $v \equiv 3 \pmod 4$ and $\lambda \equiv 0 \pmod 2$, say $v = 4t - 3$. Consider the blocks:

$[0, 1, 2t+2, 2t+1]_C, [0, 2, 2t+3, 2t+1]_C$,
$[0, 1+2i, 4t+2, 4t+1-2i]_C$ for $i = 0, 1, \ldots, t-1$,
$[0, 3+2i, 4t+2, 4t-1-2i]_C$ for $i = 0, 1, \ldots, t-2$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. For general λ even, we take $\lambda/2$ copies of these blocks. ∎

In a cyclic X–decomposition of D_v^λ, the length of the orbit of every block is v. Therefore a necessary condition for the existence of such a decomposition is that $\lambda(v-1) \equiv 0 \pmod 4$.

Theorem 2.2 *A cyclic X–decomposition of D_v^λ exists if and only if*

1. $v \equiv 0$ *or* 2 *(mod* 4*) and* $\lambda \equiv 0$ *(mod* 4*), or*

2. $v \equiv 1$ *(mod* 4*), except for* $v = 5$ *and* $\lambda = 1$*, or*

3. $v \equiv 3$ *(mod* 4*) and* $\lambda \equiv 0$ *(mod* 2*).*

Proof. Of course, no X−decomposition of D_5 exists [9]. This combined with the fact that $\lambda(v-1) \equiv 0$ (mod 4) gives the necessary conditions. We show sufficiency in four cases.

Case 1. Suppose $v \equiv 0$ (mod 4) and $\lambda \equiv 0$ (mod 4), say $v = 4t$. Consider the blocks:

$[0, 2t+1, 2, 1]_X$,
$[0, 4t-2-2i, 4t-1, 1+4i]_X$ for $i = 0, 1, 2, \ldots, t-1$,
$[0, 1+2i, 4t-1, 2+4i]_X$ for $i = 0, 1, 2, \ldots, t-1$,
$[0, 2+2i, 1, 4+4i]_X$ for $i = 0, 1, 2, \ldots, t-2$, and
$[0, 2+2i, 1, 5+4i]_X$ for $i = 0, 1, 2, \ldots, t-2$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 4$. For general $\lambda \equiv 0$ (mod 4), we take $\lambda/4$ copies of these blocks.

Case 2. Suppose $v \equiv 1$ (mod 4), $(v, \lambda) \neq (5, 1)$. For such v with $v > 5$, there exists a cyclic X−decomposition of D_v [9]. For general λ, we take λ copies of the blocks of such a decomposition. For $v = 5$ and $\lambda = 2$, consider the blocks $[0, 3, 4, 1]_X$ and $[0, 2, 1, 4]_X$. For $v = 5$ and $\lambda = 3$, consider the blocks $[0, 1, 2, 4]_X$, $[0, 2, 1, 4]_X$, and $[0, 1, 4, 2]_X$. In both cases, these blocks form a collection of base blocks for a cyclic X−decomposition of D_5^λ. For general $\lambda > 2$ even, we take $\lambda/2$ copies of the base blocks for the $\lambda = 2$ case. For general $\lambda > 3$ odd, we take one copy of the base blocks for the $v = 5$, $\lambda = 3$ decomposition and $(\lambda-3)/2$ copies of the base blocks for the $v = 5$, $\lambda = 2$ decomposition.

Case 3. Suppose $v \equiv 2$ (mod 4) and $\lambda \equiv 0$ (mod 4), say $v = 4t+2$. Consider the blocks:

$[0, 2t+1, 2t, 4t+1]_X$,
$2 \times [0, 1+2i, 4t+1, 4t-1-2i]_X$ for $i = 0, 1, 2, \ldots, t-1$, and
$2 \times [0, 4t+1-2i, 1, 2+2i]_X$ for $i = 0, 1, 2, \ldots, t-1$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 4$. For general $\lambda \equiv 0$ (mod 4), we take $\lambda/4$ copies of these blocks.

Case 4. Suppose $v \equiv 3 \pmod 4$ and $\lambda \equiv 0 \pmod 2$, say $v = 4t + 3$. Consider the blocks:

$[0, 1, 3, 6]_X, [0, 1, 4t + 2, 4]_X, [0, 2, 1, 2t + 2]_X,$
$[0, 3, 4t + 2, 4t + 1]_X$ (omit if $t = 1$),
$[0, 4 + 2i, 1, 4t - 2i]_X$ for $i = 0, 1, 2, \ldots, t - 2,$
$[0, 5 + 2i, 4t + 2, 2t + 1 - 2i]_X$ for $i = 0, 1, 2, \ldots, t - 3.$

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. For general λ even, we take $\lambda/2$ copies of these blocks. ∎

As with X–decompositions, the orbit of every block in a cyclic Y–decomposition of D_v^λ is of length v and we need $\lambda(v - 1) \equiv 0 \pmod 4$.

Theorem 2.3 *A cyclic Y–decomposition of D_v^λ exists if and only if*

1. $v \equiv 0$ *or* $2 \pmod 4$ *and* $\lambda \equiv 0 \pmod 4$, *or*
2. $v \equiv 1 \pmod 4$, *except for* $(v, \lambda) = (5, 1)$ *and* $(v, \lambda) = (5, 3)$, *or*
3. $v \equiv 3 \pmod 4$ *and* $\lambda \equiv 0 \pmod 2$.

Proof. Of course, no Y–decomposition of D_5 exists [9]. One can easily verify by exhaustion that no cyclic Y–decomposition of D_5^3 exists. The remaining necessary conditions follow from the fact that $\lambda(v - 1) \equiv 0 \pmod 4$. We show sufficiency in four cases.

Case 1. Suppose $v \equiv 0 \pmod 4$ and $\lambda \equiv 0 \pmod 4$, say $v = 4t$. First, with $v = 4$ consider $[0, 1, 3, 2]_Y$, $[0, 1, 2, 3]_Y$, and $[0, 2, 1, 3]_Y$. For $v > 4$ consider the blocks:

$[0, 1, 4, 3]_Y, [0, 1, 3, 2]_Y,$
$[0, 4t - 2 - 2i, 4t - 3 - 4i, 4t - 1 - 2i]_Y$ for $i = 0, 1, 2, \ldots, 2t - 2$, and
$[0, 4t - 2 - 2i, 4t - 3 - 4i, 4t - 1 - 2i]_Y$ for $i = 0, 1, 2, \ldots 2t - 3.$

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 4$. For general $\lambda \equiv 0 \pmod 4$, we take $\lambda/4$ copies of these blocks.

Case 2. Suppose $v \equiv 1 \pmod 4$ and $\lambda \notin \{1,3\}$ when $v = 5$. For such v with $v > 5$, there exists a cyclic $Y-$decomposition of D_v [9]. For general λ, we take λ copies of the blocks of such a decomposition. For $v = 5$ and $\lambda = 2$, consider the blocks $[0,1,3,2]_Y$ and $[0,3,2,4]_Y$. For $v = 5$ and $\lambda = 5$, consider the blocks $2 \times [0,1,3,4]_Y$, $[0,2,1,3]_Y$, $[0,1,4,3]_Y$, and $[0,1,4,2]_Y$. For $v = 5$ and $\lambda > 5$ we take $\lambda/2$ copies of the relevant base blocks when λ is even, and when λ is odd we take one copy of the base blocks for the decomposition with $\lambda = 5$ and $(\lambda - 5)/2$ copies of the base blocks for the decomposition with $\lambda = 2$.

Case 3. Suppose $v \equiv 2 \pmod 4$ and $\lambda \equiv 0 \pmod 4$, say $v = 4t + 2$. Consider the blocks:

$$[0,1,3,2]_Y,\ [0,1,4,3]_Y,\ [0,2,5,3]_Y, \text{ and}$$
$$2 \times [0, 4t+1-2i, 4t-1-4i, 4t-2i]_Y \text{ for } i = 0,1,2,\ldots,2t-2.$$

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 4$. For general $\lambda \equiv 0 \pmod 4$, we take $\lambda/4$ copies of these blocks.

Case 4. Suppose $v \equiv 3 \pmod 4$ and $\lambda \equiv 0 \pmod 2$, say $v = 4t + 3$. Consider the blocks:

$$[0,1,2t+3,2t+2]_Y,$$
$$[0,2+2i,1,4t+2-2i]_Y \text{ for } i = 0,1,2,\ldots,t-1,$$
$$[0,3+2i,1,4t+1-2i]_Y \text{ for } i = 0,1,2,\ldots,t-1.$$

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. For general λ even, we take $\lambda/2$ copies of these blocks. ∎

In a collection of base blocks for a cyclic $Z-$decomposition of D_v^λ, the differences associated with a copy of Z must satisfy one of the following conditions:

D. There are four (not necessarily distinct) differences a_i, b_i, c_i, d_i such that $a_i + b_i \equiv c_i + d_i \pmod v$ and the length of the orbit of the block is v.

E. There are two distinct differences a_i, b_i such that $a_i + b_i \equiv 0 \pmod{v/2}$ and the length of the orbit of the block is $v/2$.

We have the following necessary conditions.

Lemma 2.2 *If a cyclic $Z-$decomposition of D_v^λ exists, then*

1. $v \equiv 0$ *or* 3 *(mod 4) and* $\lambda \equiv 0$ *(mod 2), or*

2. $v \equiv 1$ *(mod 4), or*

3. $v \equiv 2$ *(mod 4) and* $\lambda \equiv 0$ *(mod 4).*

Proof. As with all of these decompositions, we need $\lambda v(v-1) \equiv 0$ (mod 4).

First, consider a $Z-$decomposition of D_v^λ where $v \equiv 0$ (mod 4) and $\lambda \equiv 1$ (mod 2) (regardless of the automorphism which it admits). The out-degree of each vertex of D_v^λ is $\lambda(v-1) \equiv 1$ (mod 2). Since the out-degree of each vertex of Z is even, there exists no such decomposition.

Suppose there exists a cyclic $Z-$decomposition of D_v^λ with $v \equiv 2$ (mod 4) and $\lambda \equiv 2$ (mod 4). Then it is necessary to partition the difference multiset $\lambda \times \{1, 2, \ldots, v-1\}$ into differences associated with base blocks in such a way as to satisfy conditions D and E. The total number of differences is $\lambda(v-1) \equiv 2$ (mod 4). Since there are four differences associated with each base block which satisfies condition D, there must be an odd number of base blocks satisfying condition E. A base block satisfying condition E must have one odd associated difference and one even associated difference. Therefore all base blocks satisfying condition E must have an odd number of even differences associated with them. Now each base block satisfying condition D has an even number of even differences associated with it. However, this means that a collection of base blocks for such a decomposition must have an associated collection of differences which consists of an odd number of even differences. Since λ is even, though, the total collection of differences has an even number of even differences. This contradiction implies that no such decomposition exists. ∎

We now show that these necessary conditions are in fact sufficient.

Theorem 2.4 *A cyclic* $Z-$*decomposition of* D_v^λ *exists if and only if*

1. $v \equiv 0$ *or* 3 *(mod 4) and* $\lambda \equiv 0$ *(mod 2), or*

2. $v \equiv 1$ *(mod 4).*

3. $v \equiv 2$ *(mod 4) and* $\lambda \equiv 0$ *(mod 4).*

Proof. The necessary conditions follow from Lemma 2.2. We show sufficiency in four cases.

Case 1. Suppose $v \equiv 0$ (mod 4) and $\lambda \equiv 0$ (mod 2), say $v = 4t$. Consider the following blocks:

$[0, 1, 2t, 2t+1]_Z$,
$[0, 2+2i, 1, 4t-1-2i]_Z$ for $i = 0, 1, 2, \ldots, t-1$, and
$[0, 3+2i, 1, 4t-1-2i]_Z$ for $i = 0, 1, 2, \ldots, t-2$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. For general λ even, we take $\lambda/2$ copies of these blocks.

Case 2. Suppose $v \equiv 1 \pmod 4$. A cyclic 4−circuit decomposition of D_v exists [7]. For general λ, we take λ copies of the blocks of such a decomposition.

Case 3. Suppose $v \equiv 2 \pmod 4$ and $\lambda \equiv 0 \pmod 4$, say $v = 4t+2$. Consider the blocks:

$[0, 4t+1, 2t, 2t+1]_Z$,
$2 \times [0, 2+2i, 1, 4t+1-2i]_Z$ for $i = 0, 1, 2, \ldots, t-1$,
$[0, 2+2i, 1, 3+2i]_Z$ for $i = 0, 1, 2, \ldots, t-1$, and
$[0, 4t+1-2i, 1, 4t-2i]_Z$ for $i = 0, 1, 2, \ldots, t-1$.

This collection of blocks forms a collection of base blocks for th desired decomposition when $\lambda = 4$. For general $\lambda \equiv 0 \pmod 4$, w take $\lambda/4$ copies of these blocks.

Case 4. Suppose $v \equiv 3 \pmod 4$ and $\lambda \equiv 0 \pmod 2$, say $v = 4t+$ Consider the following blocks:

$[0, 2, 1, 2t+2]_Z$,
$[0, 2+2i, 1, 4t+2-2i]_Z$ for $i = 0, 1, 2, \ldots, t-1$, and
$[0, 4+2i, 1, 4t+2-2i]_Z$ for $i = 0, 1, 2, \ldots, t-1$.

This collection of blocks forms a collection of base blocks for t desired decomposition when $\lambda = 2$. For general λ even, we take λ, copies of these blocks.

Theorems 2.1 through 2.4 give necessary and sufficient conditions the existence of cyclic decompositions of D_v^λ into each of the orientatio of a 4−cycle.

3 Rotational Decompositions

Throughout this section, we let the vertex set of D_v^λ be $\{\infty\} \bigcup \mathbf{Z}_{v-1}$ and suppose that a rotational decomposition admits the automorphism $\beta = (\infty)(0, 1, 2, \ldots, v-2)$.

Theorem 3.1 *A rotational* 4*−circuit decomposition of* D_v^λ *exists if and only if*

1. $v \equiv 0 \pmod 4$ *and* $\lambda > 1$*, except* $v = 4$ *and* λ *odd, or*
2. $v \equiv 1 \pmod 4$*, or*
3. $v \equiv 2$ *or* $3 \pmod 4$ *and* $\lambda \equiv 0 \pmod 2$*.*

Proof. First, it is necessary that $\lambda v(v-1) \equiv 0 \pmod 4$. Pennisi [14] has shown that a rotational 4−circuit decomposition of D_v exists if and only if $v \equiv 1 \pmod 4$. It is shown in the appendix that for $v = 4$, λ must be even. These facts establish the necessary conditions. We now establish sufficiency in four cases.

Case 1. Suppose $v \equiv 0 \pmod 4$, $\lambda > 1$ and $v \neq 4$ if λ is odd. Say $v = 4t$. Then consider the blocks:

$[0, \infty, 1, 2t]_C, [0, \infty, 4t-2, 2t-1]_C$, and
$2 \times [0, 1+2i, 3+4i, 2+2i]_C$ for $i = 0, 1, 2, \ldots, t-2$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. Next, consider the blocks:

$[0, \infty, 2t-1, 2t]_C$, $[0, \infty, 2t+1, 2t]_C$, $[0, \infty, 4t-2, 2t-1]_C$,
$2 \times [0, 1+2i, 3+4i, 2+2i]_C$ for $i = 0, 1, 2, \ldots, t-2$, and
$[0, 2+2i, 4t-2, 4t-4-2i]_C$ for $i = 0, 1, 2, \ldots, t-2$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 3$. For general $\lambda > 1$ even, we take $\lambda/2$ copies of the base blocks from the $\lambda = 2$ case. For general $\lambda > 1$ odd we take 1 copy of the base blocks from the $\lambda = 3$ case and $(\lambda - 3)/2$ copies of the base blocks from the $\lambda = 2$ case.

Case 2. Suppose $v \equiv 1 \pmod 4$. Then there exists a rotational 4−circuit decomposition of D_v [14]. For general λ, we take λ copies of the blocks of such a decomposition.

Case 3. Suppose $v \equiv 2 \pmod 4$ and $\lambda \equiv 0 \pmod 2$, say $v = 4t + 2$. Consider the blocks:

$[0, \infty, 4t-2, 4t-1]_C$, $[0, \infty, 3, 1]_C$,
$[0, 1+2i, 3+4i, 2+2i]_C$ for $i = 0, 1, 2, \ldots, t-1$, and
$[0, 3+2i, 7+4i, 3+2i]_C$ for $0, 1, 2, \ldots, t-2$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. For general $\lambda > 1$ even, we take $\lambda/2$ copies of these base blocks.

Case 4. Suppose $v \equiv 3 \pmod 4$ and $\lambda \equiv 0 \pmod 2$, say $v = 4t + 3$. Consider the blocks:

$[0, \infty, 1, 2t]_C$, $[0, 4t+1, \infty, 2t-1]_C$, $[0, 1, 2t+1, 2t+2]_C$, $[0, 2t, 4t+1, 2t+1]_C$,
$[0, 1+2i, 3+4i, 2+2i]_C$ for $i = 0, 1, 2, \ldots, t-2$, and
$[0, 2t-2-2i, 4t-3-4i, 2t-1-2i]_C$ for $i = 0, 1, 2, \ldots, t-2$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. For general $\lambda > 1$ even, we take $\lambda/2$ copies of these base blocks. ∎

In a rotational X−decomposition of D_v^λ, the length of the orbit of every block is $v - 1$. Therefore a necessary condition for the existence of such a decomposition is that $\lambda v \equiv 0 \pmod 4$.

Theorem 3.2 *A rotational X−decomposition of D_v^λ exists if and only if*

1. $v \equiv 0$ *(mod 4),*
2. $v \equiv 1$ *or* 3 *(mod 4) and* $\lambda \equiv 0$ *(mod 4), or*
3. $v \equiv 2$ *(mod 4) and* $\lambda \equiv 0$ *(mod 2).*

Proof. The necessary conditions follow from the fact that $\lambda v \equiv 0 \pmod 4$. We establish sufficiency in four cases.

Case 1. Suppose $v \equiv 0 \pmod 4$. Then there exists a rotational X−decomposition of D_v [7]. For general λ, we take λ copies of the blocks of such a decomposition.

Case 2. Suppose $v \equiv 1 \pmod 4$ and $\lambda \equiv 0 \pmod 4$, say $v = 4t + 1$. Consider the blocks:

$2 \times [0, 2t+1, \infty, 2t]_X$, $2 \times [0, 2t+1, \infty, 2t-1]_X$,
$[0, 4t-2-2i, 4t-1, 1+2i]_X$ for $i = 0, 1, 2, \ldots, t-1$,
$[0, 4t-2-2i, 4t-1, 1+2i]_X$ for $i = 0, 1, 2, \ldots, t-2$, and
$2 \times [0, 2+2i, 1, 4t-1-2i]_X$ for $i = 0, 1, 2, \ldots, t-2$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 4$. For general $\lambda \equiv 0 \pmod 4$, we take $\lambda/4$ copies of these base blocks.

Case 3. Suppose $v \equiv 2 \pmod 4$ and $\lambda \equiv 0 \pmod 2$, say $v = 4t + 2$. Consider the blocks:

$[0, 2t, \infty, 1]_X$, $[0, 2t, \infty, 4t]_X$,
$[0, 1+2i, 4t, 4t-1-2i]_X$ for $i = 0, 1, 2, \ldots, t-1$, and
$[0, 2+2i, 5+4i, 2+2i]_X$ for $i = 0, 1, 2, \ldots, t-2$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. For general λ even, we take $\lambda/2$ copies of these base blocks.

Case 4. Suppose $v \equiv 3 \pmod 4$ and $\lambda \equiv 0 \pmod 4$, say $v = 4t + 3$. Consider the blocks:

$2 \times [0, 2t+3, \infty, 2t+1]_X$ $[0, 2t, \infty, 2t+1]_X$, $[0, 2t+2, \infty, 2t+1]_X$,
$2 \times [0, 4t-2i, 4t+1, 1+2i]_X$ for $i = 0, 1, 2, \ldots, t-1$,
$[0, 2+2i, 1, 4t+1-2i]_X$ for $i = 0, 1, 2, \ldots, t-1$, and
$[0, 2+2i, 1, 4t-1-2i]_X$ for $i = 0, 1, 2, \ldots, t-2$.

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 4$. For general $\lambda \equiv 0 \pmod 4$, we take $\lambda/4$ copies of these base blocks. ∎

As with X–decompositions, the orbit of every block in a rotational Y–decomposition of D_v^λ is of length $v - 1$ and we need $\lambda v \equiv 0 \pmod 4$.

Theorem 3.3 *A rotational Y–decomposition of D_v^λ exists if and only if*

1. $v \equiv 0$ *(mod 4), except $v = 4$ and λ odd, or*

2. $v \equiv 1$ *or* 3 *(mod 4) and $\lambda \equiv 0$ (mod 4), or*

3. $v \equiv 2$ *(mod 4) and $\lambda \equiv 0$ (mod 2).*

Proof. See the appendix for an argument that no Y-decomposition of D_4^λ exists for λ odd. This fact, combined with the condition $\lambda v \equiv 0 \pmod 4$ give the necessary conditions. We show these conditions are sufficient in five cases.

Case 1. Suppose $v = 4$. Consider the blocks $[0,1,2,\infty]_Y$ and $[0,2,1,\infty]_Y$. This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. For general λ even, we take $\lambda/2$ copies of these base blocks.

Case 2. Suppose $v \equiv 0 \pmod 4$, $v \neq 4$. Then there exists a rotational Y-decomposition of D_v [7]. For general λ, we take λ copies of the blocks of such a decomposition.

Case 3. Suppose $v \equiv 1 \pmod 4$ and $\lambda \equiv 0 \pmod 4$, say $v = 4t+1$. Consider the blocks:

$$2 \times [0, 2t, 1, \infty]_Y,\ 2 \times [0, 2t, 4t-1, \infty]_Y,\ [0, 1, 2t+2, 2t+1]_Y,$$
$$[0, 1+2i, 3+4i, 2+2i]_Y \text{ for } i = 0, 1, 2, \ldots, t-2,$$
$$[0, 2+2i, 5+4i, 3+2i]_Y \text{ for } i = 0, 1, 2, \ldots, t-2, \text{ and}$$
$$2 \times [0, 4t-1-2i, 4t-3-4i, 4t-2-2i]_Y \text{ for } i = 0, 1, 2, \ldots, t-2.$$

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 4$. For general $\lambda \equiv 0 \pmod 4$, we take $\lambda/4$ copies of these base blocks.

Case 4. Suppose $v \equiv 2 \pmod 4$ and $\lambda \equiv 0 \pmod 2$, say $v = 4t+2$. Consider the blocks:

$$2 \times [0, 4t, 4t-2, \infty]_Y, \text{ and}$$
$$[0, 1+2i, 3+4i, 2+2i]_Y \text{ for } i = 0, 1, 2, \ldots, 2t-2.$$

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. For general λ even, we take $\lambda/2$ copies of these base blocks.

Case 5. Suppose $v \equiv 3 \pmod 4$ and $\lambda \equiv 0 \pmod 4$, say $v = 4t+3$. Consider the blocks:

$$2 \times [0, 2t+1, 1, \infty]_Y,\ 2 \times [0, 2t+1, 2, \infty]_Y,$$
$$2 \times [0, 1+2i, 3+4i, 2+2i]_Y \text{ for } i = 0, 1, 2, \ldots, t-1,$$
$$[0, 4t-2i, 4t-1-4i, 4t+1-2i]_Y \text{ for } i = 0, 1, 2, \ldots, t-1, \text{ and}$$
$$[0, 4t-2i, 4t-1-4i, 4t+1-2i]_Y \text{ for } i = 0, 1, 2, \ldots, t-2.$$

This collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 4$. For general $\lambda \equiv 0 \pmod 4$, we take $\lambda/4$ copies of these base blocks. ∎

In a rotational Z–decomposition of D_v^λ, the arc $(\infty, 0)$ occurs λ times. Now since each vertex of Z is of even out-degree, the arc $(\infty, 0)$ must occur an even number of times in such a decomposition. Therefore, λ must be even. In fact, as we see in the following theorem, this condition is sufficient.

Theorem 3.4 *A rotational Z–decomposition of D_v^λ exists if and only if λ is even.*

Proof. We consider four cases.

Case 1. Suppose $v \equiv 0 \pmod 4$ and $\lambda \equiv 0 \pmod 2$, say $v = 4t$. Consider the blocks:

$$[0, 2t, \infty, 2t-1]_Z,\ [0, 2t, 1, \infty]_Z,$$
$$2 \times [0, 4t-2-2i, 1, 2+2i]_Z \text{ for } i = 0, 1, 2, \dots, t-2.$$

Case 2. Suppose $v \equiv 1 \pmod 4$ and $\lambda \equiv 0 \pmod 2$, say $v = 4t+1$. Consider the blocks:

$$[0, 2t, \infty, 2t+1]_Z,\ [0, 2t, 1, \infty]_Z,\ [0, 1, 2t, 2t+1]_Z,$$
$$[0, 4t-1-2i, 1, 2+2i]_Z \text{ for } i = 0, 1, 2, \dots, t-2,$$
$$[0, 4t-1-2i, 1, 3+2i]_Z \text{ for } i = 0, 1, 2, \dots, t-2.$$

Case 3. Suppose $v \equiv 2 \pmod 4$ and $\lambda \equiv 0 \pmod 2$, say $v = 4t+2$. Consider the blocks:

$$[0, 2t-1, \infty, 4t]_Z,\ [0, 2t+1, 2t+2, \infty]_Z,$$
$$[0, 4t-1-2i, 1, 2+2i]_Z \text{ for } i = 0, 1, 2, \dots, t-1,$$
$$[0, 4t-1-2i, 1, 2+2i]_Z \text{ for } i = 0, 1, 2, \dots, t-2.$$

Case 4. Suppose $v \equiv 3 \pmod 4$ and $\lambda \equiv 0 \pmod 2$, say $v = 4t+3$. Consider the blocks:

$$[0, 2t, \infty, 2t+1]_Z,\ [0, 2t+3, 2, \infty]_Z,\ [0, 1, 2t+1, 2t+2]_Z,$$
$$[0, 4t+1-2i, 1, 2+2i]_Z \text{ for } i = 0, 1, 2, \dots, t-1,$$
$$[0, 4t+1-2i, 1, 3+2i]_Z \text{ for } i = 0, 1, 2, \dots, t-2.$$

In each case, the collection of blocks forms a collection of base blocks for the desired decomposition when $\lambda = 2$. For general λ even, we take $\lambda/2$ copies of these base blocks. ∎

Theorems 3.1 through 3.4 give necessary and sufficient conditions for the existence of rotational decompositions of D_v^λ into each of the orientations of a 4–cycle.

4 Conclusion

We now use the results of sections 2 and 3 to establish the existence of decompositions of D_v^λ into each of the orientations of a 4–cycle.

Theorem 4.1 *A 4–circuit decomposition of D_v^λ exists if and only if $\lambda v(v-1) \equiv 0 \pmod 4$, except $v = 4$ and λ odd.*

Proof. The condition $\lambda v(v-1) \equiv 0 \pmod 4$ is obvious. The proof that no such decomposition exists for $v = 4$ and λ odd is given in the appendix. The constructions for $\lambda = 1$ are given in [15]. The constructions for $\lambda > 1$ are covered in Theorems 2.1 and 3.1. ∎

Theorem 4.2 *An X–decomposition of D_v^λ exists if and only if $\lambda v(v-1) \equiv 0 \pmod 4$, except $v = 5$ and $\lambda = 1$.*

Proof. The condition $\lambda v(v-1) \equiv 0 \pmod 4$ is obvious. The case of $\lambda = 1$ is covered in [9]. The constructions for $\lambda > 1$ are covered in Theorems 2.2 and 3.2. ∎

Theorem 4.3 *A Y–decomposition of D_v^λ exists if and only if $\lambda v(v-1) \equiv 0 \pmod 4$, except ($v = 4$ and λ odd) and ($v = 5$ and $\lambda = 1$).*

Proof. The condition $\lambda v(v-1) \equiv 0 \pmod 4$ is obvious. The fact that no such decomposition exists for $v = 4$ and λ odd is discussed in the appendix. The case of $\lambda = 1$ is covered in [9]. The constructions for $\lambda > 1$ (except for $v = 5$ and $\lambda = 3$) are covered in Theorems 2.3 and 3.3. A Y–decomposition of D_5^3 is given by: $[0,1,3,2]_Y$, $[3,4,2,1]_Y$, $[3,2,4,0]_Y$, $[1,0,3,4]_Y$, $[2,1,4,0]_Y$, $[0,4,3,2]_Y$, $[3,1,2,4]_Y$, $[3,2,1,0]_Y$, $[4,0,3,1]_Y$, $[2,4,1,0]_Y$, $[4,1,3,2]_Y$, $[3,0,2,1]_Y$, $[3,2,0,4]_Y$, $[1,4,3,0]_Y$, $[2,1,0,4]_Y$. ∎

Theorem 4.4 *A Z–decomposition of D_v^λ exists if and only if $\lambda v(v-1) \equiv 0 \pmod 4$, except $v \equiv 0 \pmod 4$ and λ odd.*

Proof. The condition $\lambda v(v-1) \equiv 0 \pmod 4$ is obvious. The proof that no such decomposition for $v \equiv 0 \pmod 4$ and λ odd is given in the proof of

Lemma 2.2. The constructions for $\lambda = 1$ are given in [9]. The constructions for $\lambda > 1$ are covered in Theorems 2.4 and 3.4. ∎

Theorems 4.1 through 4.4 give necessary and sufficient conditions for decompositions of D_v^λ into each of the orientations of a 4−cycle.

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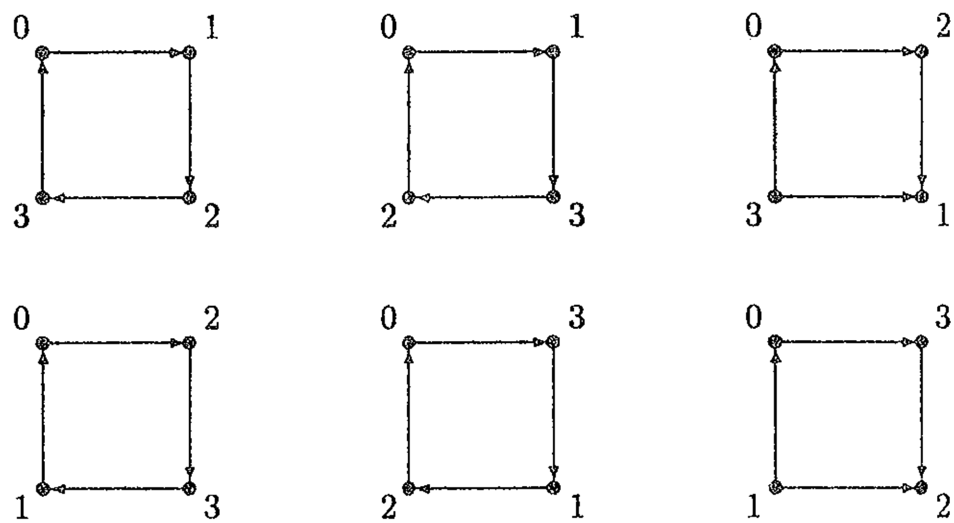
Appendix

If we consider C_4−decompositions of D_4^λ, then there are six possible blocks:



We denote the number of these blocks in such a decomposition as n_i for $i = 1, 2, \ldots, 6$, respectively. Considering the number of arcs in this decomposition, we get 12 equations in the 6 unknowns:

arc													
01:	n_1	+	n_2	+	0	+	0	+	0	+	0	=	λ
02:	0	+	0	+	n_3	+	n_4	+	0	+	0	=	λ
03:	0	+	0	+	0	+	0	+	n_5	+	n_6	=	λ
10:	0	+	0	+	0	+	n_4	+	0	+	n_6	=	λ
12:	n_1	+	0	+	0	+	0	+	n_5	+	0	=	λ
13:	0	+	n_2	+	n_3	+	0	+	0	+	0	=	λ
20:	0	+	n_2	+	0	+	0	+	n_5	+	0	=	λ
21:	0	+	0	+	n_3	+	0	+	0	+	n_6	=	λ
23:	n_1	+	0	+	0	+	n_4	+	0	+	0	=	λ
30:	n_1	+	0	+	n_3	+	0	+	0	+	0	=	λ
31:	0	+	0	+	0	+	n_4	+	n_5	+	0	=	λ
32:	0	+	n_2	+	0	+	0	+	0	+	n_6	=	λ

Adding the equations for arcs $(0,1)$ and $(1,3)$, we get the condition: $n_1 + 2n_2 + n_3 = 2\lambda$. Subtracting the equation for arc $(3,0)$ from *this* equation, gives: $2n_2 = \lambda$. Therefore, a necessary condition for such a decomposition to exist is that λ is even.

A similar analysis of Y−decompositions of D_4^λ reveals that λ must be even in that case as well. This time, however, we must deal with 12 equations in 12 unknowns.