Decomposing and Packing the Complete Graph with Osculating 4-Cycles[1]

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Abstract. The *osculating* 4*-cycles* graph, denoted OC_4, consists of two 4-cycles with exactly one vertex in common. Necessary and sufficient conditions are given for the existence of a decomposition of the complete graph into OC_4's. Necessary and sufficient conditions are also presented for maximal packings of the complete graph with OC_4's.

1 Introduction

A *decomposition* of a simple graph G into isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $E(g_i) \bigcap E(g_j) = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{n} E(g_i) = E(G)$, where $V(G)$ is the vertex set of graph G and $E(G)$ is the edge set of graph G. We will refer to such a decomposition as a "g-decomposition of G." In the event that a g-decomposition of G does not exist, we can ask the question "How close can we get to a g-decomposition of G?" One approach is the idea of a "packing."

A *maximal packing* of a simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $E(g_i) \bigcap E(g_j) = \emptyset$ if $i \neq j$, $\bigcup_{i=1}^{n} g_i \subset G$, and

$$\left| E(G) \setminus \bigcup_{i=1}^{n} E(g_i) \right|$$

is minimal. Packings of complete graphs have been studied, for example, for the graph g a 3-cycle [8], a 4-cycle [9], K_4 [2], and a 6-cycle [5, 6].

Consider the graph

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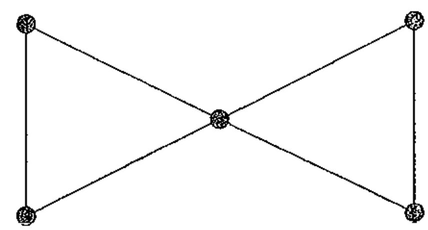
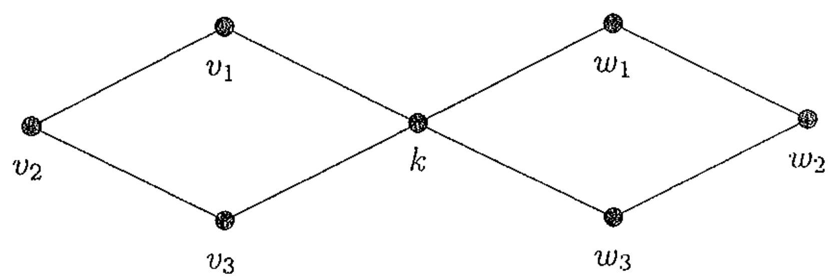
$OC_3 =$ 

which we call *osculating 3-cycles* (the terms *bowtie* [1] and 2-*windmill* [4] have also been used). Horák and Rosa [4] solve the decomposition and packing problems of complete graphs on v vertices, K_v, with OC_3's in the cases $v \equiv 1$ or 3 (mod 6) (they actually address decompositions and packing problems of Steiner triple systems). In this paper, we concentrate on the graph

$OC_4 =$ 

which we call *osculating 4-cycles.* With the vertices as labeled, we denote this graph as $([k, v_1, v_2, v_3], [k, w_1, w_2, w_3])$. The purpose of this paper is to give necessary and sufficient conditions for decompositions and maximal packings of complete graphs with OC_4's. In each case, we will give direct constructions of an optimal set of OC_4's.

2 Decompositions

For an OC_4-decomposition of K_v, it is clear that we need $|E(K_v)| \equiv 0$ (mod 8); that is, $v \equiv 0$ or 1 (mod 16). Also, since each vertex of OC_4 is of even degree, we need v odd. Therefore, a necessary condition for an OC_4-decomposition of K_v is $v \equiv 1$ (mod 16). We use a simple cyclic construction to show this necessary condition is in fact sufficient. Throughout this paper, we take the vertex set of K_v as $\{0, 1, \ldots, v-1\}$.

Theorem 2.1 *An OC_4-decomposition of K_v exists if and only if $v \equiv 1$ (mod 16).*

Proof. We only need to establish sufficiency. Consider the set

$$\{([j, 1+8i+j, 5+16i+j, 2+8i+j], [j, 8+8i+j, 13+16i+j, 7+8i+j])$$
$$\text{for } i = 0, 1, \ldots, (v-17)/16 \text{ and } j = 0, 1, \ldots, v-1\}$$

(where the labels of the vertices are reduced modulo v). This set is an OC_4-decomposition of K_v. ∎

3 Packings

In a maximal packing of G with copies of g, we call the graph induced by $E(G) \setminus \bigcup_{i=1}^{n} E(g_i)$ the *leave*, L, of the packing. In this section we give necessary and sufficient conditions for a maximal packing of K_n with OC_4's. We start with some initial results.

Lemma 3.1 *An OC_4-decomposition of $K_{m,n}$ exists if and only if $n \equiv 0$ (mod 2), $m \equiv 0$ (mod 4), $m \geq 4$, and $n \geq 4$.*

Proof. Since the degree of each vertex of OC_4 is even, in such a decomposition it is necessary that the degree of each vertex of $K_{m,n}$ must be even. Therefore $m \equiv n \equiv 0$ (mod 2) is necessary. Since $|E(OC_4)| = 8$, we also need $|E(K_{m,n})| \equiv 0$ (mod 8), and therefore (without loss of generality) $m \equiv 0$ (mod 4) is necessary. Finally, since OC_4 is a bipartite graph with the vertex set of one part having cardinality 3 and the vertex set of the other part having cardinality 4, we need both m and n to be greater than or equal to 4.

Now for sufficiency, suppose the partite sets of $K_{m,n}$ are $\{1_1, 2_1, \ldots, m_1\}$ and $\{1_2, 2_2, \ldots, n_2\}$.

Case 1. Suppose $m \equiv n \equiv 0$ (mod 4). Consider the set:

$$\{([(2+4i)_1, (1+4j)_2, (1+4i)_1, (2+4j)_2][(2+i)_1, (3+4j)_2, (3+3i)_1, (4+4j)_2])$$
$$\text{for } i = 0, 1, \ldots, m/4-1 \text{ and } j = 0, 1, \ldots, n/4-1\}.$$

Case 2. Suppose $m \equiv 0$ (mod 4) and $n \equiv 2$ (mod 4), $n \geq 6$. Consider the set:

$$\{([2_2, (1+4j)_1, 1_2, (2+4j)_1][2_2, (3+4j)_1, 3_2, (4+4j)_1]),$$
$$([6_2, (4+4j)_1, 5_2, (3+4j)_1][6_2, (2+4j)_1, 3_2, (1+4j)_1]),$$
$$([4_2, (1+4j)_1, 5_2, (2+4j)_1][4_2, (3+4j)_1, 1_2, (4+4j)_1])$$
$$\text{for } j = 0, 1, \ldots, m/4-1\} \bigcup$$

$$\{([(8+4i)_2,(1+4j)_1,(7+4i)_2,(2+4j)_1],[(8+4i)_2,(3+4j)_1,(9+4i)_2,(4+4j)_1]),$$
$$([(10+4i)_2,(1+4j)_1,(9+4i)_2,(2+4j)_1],[(10+4i)_2,(4+4j)_1,(7+4i)_2,(3+4j)_1])$$
$$\text{for } i = 0, 1, \ldots, (n-10)/4 \text{ and } j = 0, 1, \ldots, m/4 - 1\}.$$

In each case, the given set is an OC_4-decomposition of $K_{m,n}$. ∎

Lemma 3.2 *A maximal packing of $K_{m,n}$ where $m \equiv n \equiv 2 \pmod 4$, $m \geq 6$, and $n \geq 6$ with OC_4's has a leave $L = C_4$.*

Proof. Suppose m and n satisfy the given conditions. Then $|E(K_{m,n})| \equiv 4 \pmod 8$. Therefore a leave L with $|E(L)| = 4$ would be optimal. Also, each vertex of $K_{m,n}$ is of even degree and each vertex of OC_4 is of even degree, so if $|E(L)| = 4$ then it must be that $L = C_4$. Consider the set:

$$\{([2_1, 1_2, 1_1, 2_2][2_1, 3_2, 3_1, 4_2]), ([5_2, 5_1, 6_2, 6_1][5_2, 4_1, 2_2, 3_1]),$$
$$([1_1, 3_2, 5_1, 4_2][1_1, 5_2, 2_1, 6_2]), ([4_1, 1_2, 3_1, 6_2][4_1, 3_2, 6_1, 4_2])\}.$$

This set is an OC_4-packing of $K_{6,6}$ where the partite sets are $\{1_1, 2_1, 3_1, 4_1, 5_1, 6_1\}$ and $\{1_2, 2_2, 3_2, 4_2, 5_2, 6_2\}$ and the leave is $L = C_4 = [5_1, 1_2, 6_1, 2_2]$. With the notation of Lemma 3.1, notice that general $K_{m,n}$ can be written as

$$K_{m,n} = K_{6,6} \bigcup K_{m-6,6} \bigcup K_{6,n-6} \bigcup K_{m-6,n-6}$$

where the partite sets of $K_{6,6}$ are as above, the partite sets of $K_{m-6,6}$ are $\{7_1, 8_1, \ldots, m_1\}$ and $\{1_2, 2_2, 3_2, 4_2, 5_2, 6_2\}$, the partite sets of $K_{6,n-6}$ are $\{1_1, 2_1, 3_1, 4_1, 5_1, 6_1\}$ and $\{7_2, 8_2, \ldots, n_2\}$, and the partite sets of $K_{m-6,n-6}$ are $\{7_1, 8_1, \ldots, m_1\}$ and $\{7_2, 8_2, \ldots, n_2\}$. Since there are OC_4-decompositions of $K_{m-6,6}$, $K_{6,n-6}$, and $K_{m-6,n-6}$ by Lemma 3.1, we see that a maximal packing of $K_{m,n}$ with OC_4's has a leave of $L = C_4$. ∎

Theorem 3.1 *A maximal packing of K_v with OC_4's and leave L satisfies the following:*

1. *if $v \equiv 0$ or $2 \pmod 8$, then $|E(L)| = v/2$,*
2. *if $v \equiv 4$ or $6 \pmod 8$, then $|E(L)| = v/2 + 4$,*
3. *if $v \equiv 1, 3, 7, 9, 11$ or $13 \pmod{16}$, $v \neq 7$, then $|E(L)| = |E(K_v)| \pmod 8$,*
4. *if $v = 7$, then $|E(L)| = 13$,*
5. *if $v \equiv 5$ or $15 \pmod{16}$, then $|E(L)| = |E(K_v)| \pmod 8 + 8$.*

Proof. Theorem 2.1 takes care of $v \equiv 1 \pmod{16}$. We now consider 17 cases.

Case 1. Suppose $v \equiv 2 \pmod{16}$. Then each vertex of K_v is of odd degree. Since each vertex of OC_4 is of even degree, the leave of a packing will have each vertex of odd degree. Therefore a leave L with $|E(L)| = v/2$ would be optimal (in which case L is a perfect matching of K_v). Consider the set:

$$\{([j, 1+8i+j, 5+16i+j, 2+8i+j], [j, 8+8i+j, 13+16i+j, 7+8i+j])$$

$$\text{for } i = 0, 1, \dots, (v-18)/16 \text{ and } j = 0, 1, \dots, v-1\}$$

(where the labels of the vertices are reduced modulo v). This is a maximal packing of K_v with leave L where $E(L) = \{(i, v/2+i) \text{ for } i = 0, 1, \dots, v/2-1\}$.

Case 2. Suppose $v \equiv 3 \pmod{16}$. In this case, $|E(K_v)| \equiv 3 \pmod 8$. Therefore a leave L with $|E(L)| = 3$ would be optimal. Also, since each vertex of K_v is of even degree and each vertex of OC_4 is of even degree, if $|E(L)| = 3$ then $L = C_3$. Notice that

$$K_v = K_{v-2} \bigcup K_{v-3,2} \bigcup C_3$$

where the vertex set of K_{v-2} is $\{0, 1, \dots, v-3\}$, the partite sets of $K_{v-3,2}$ are $\{0, 1, \dots, v-4\}$ and $\{v-2, v-1\}$, and the vertex set of C_3 is $\{v-3, v-2, v-1\}$. Since an OC_4-decomposition of K_{v-2} exists by Theorem 2.1, and an OC_4-decomposition of $K_{v-3,2}$ exists by Lemma 3.1, then a maximal packing of K_v exists with leave $L = C_3$.

Case 3. Suppose $v \equiv 4 \pmod{16}$. Then, as in Case 1, each vertex of the leave must be of odd degree. The leave must therefore consist of at least $v/2$ edges. Now $|E(K_v)| \equiv 6 \pmod 8$ and $v/2 \equiv 2 \pmod 8$, so a maximal packing will have a leave L where $|E(L)| \geq v/2 + 4$. Notice that

$$K_v = K_{v-2} \bigcup K_{v-4,2} \bigcup K_2 \bigcup C_4$$

where the vertex set of K_{v-2} is $\{0, 1, \dots, v-3\}$, the partite sets of $K_{v-4,2}$ are $\{0, 1, \dots, v-5\}$ and $\{v-2, v-1\}$, the vertex set of K_2 is $\{v-2, v-1\}$, the vertex set of C_4 is $\{v-4, v-3, v-2, v-1\}$, and the edge set of C_4 is $\{(v-4, v-2), (v-4, v-1), (v-3, v-2), (v-3, v-1)\}$. First, there exists an OC_4-decomposition of $K_{v-4,2}$ by Lemma 3.1. Next, there exists a packing of K_{v-2} with leave L_1 where $|E(L_1)| = (v-2)/2$. Therefore there exists a maximal packing of K_v with leave L where $|E(L)| = |E(L_1)| + |E(K_2)| + |E(C_4)| = v/2 + 4$.

Case 4. Suppose $v \equiv 5 \pmod{16}$. Then, as in Case 2, each vertex of the leave must be of even degree. Now $|E(K_v)| \equiv 2 \pmod 8$. Clearly, each vertex of the leave cannot be of even degree if $|E(L)| = 2$. Therefore $|E(L)| \geq 10$. Notice that

$$K_v = K_{v-4} \bigcup K_{v-5,4} \bigcup K_5$$

where the vertex set of K_{v-4} is $\{0, 1, \ldots, v-5\}$, the partite sets of $K_{v-5,4}$ are $\{0, 1, \ldots, v-6\}$ and $\{v-4, v-3, v-2, v-1\}$, and the vertex set of K_5 is $\{v-5, v-4, v-3, v-2, v-1\}$. Since there exists an OC_4-decomposition of K_{v-4} by Theorem 2.1, and there exists an OC_4-decomposition of $K_{v-5,4}$ by Lemma 3.1, then there exists a maximal packing of K_v with OC_4's and $|E(L)| = |E(K_5)| = 10$.

Case 5. Suppose $v \equiv 6 \pmod{16}$. Then as in Case 1, each vertex of the leave must be of odd degree and $|E(L)| \geq v/2$. Since $|E(K_v)| \equiv 7 \pmod 8$ and $v/2 \equiv 3 \pmod 8$, it is necessary that $|E(L)| \geq v/2 + 4$. Notice that

$$K_v = K_{v-4} \bigcup K_{v-4,4} \bigcup K_4$$

where the vertex set of K_{v-4} is $\{0, 1, \ldots, v-5\}$, the partite sets of $K_{v-4,4}$ are $\{0, 1, \ldots, v-5\}$ and $\{v-4, v-3, v-2, v-1\}$, and the vertex set of K_4 is $\{v-4, v-3, v-2, v-1\}$. First, there exists an OC_4-decomposition of $K_{v-4,4}$ by Lemma 3.1. Second, there exists an OC_4-packing of K_{v-4} with leave L_1 where $|E(L_1)| = (v-4)/2$ by Case 1. Therefore there exists a maximal packing of K_v with OC_4's where $|E(L)| = |E(L_1)| + |E(K_4)| = v/2 + 4$.

Case 6. Suppose $v = 7$. Without loss of generality, $A = ([0, 1, 2, 3,], [0, 4, 5, 6])$ is in a maximal packing of K_7. Suppose there is a second OC_4 in such a packing, call it B, and that vertex a is of degree 4 in graph B. First, suppose vertex a is adjacent to vertex 0 in graph A, say (without loss of generality) that $a = 1$. Then in graph B, vertex a is adjacent to vertices 3, 4, 5, and 6. Therefore, in graph B, vertex 3 must be adjacent to either vertex 0 or vertex 2. This is impossible, since in graph A, vertex 3 is adjacent to both vertices 0 and 2. Second, suppose vertex a is not adjacent to vertex 0 in graph A, say (without loss of generality) that $a = 2$. Then in graph B, vertex a is adjacent to vertices 0, 4, 5, and 6. Therefore in graph B, vertex 0 must be adjacent to either vertex 1 or vertex 3. This is impossible since in graph A, vertex 0 is adjacent to both vertices 1 and 3. Therefore, there is only one OC_4 in a maximal packing of K_7 and the leave L of such a packing satisfies $|E(L)| = 13$.

Case 7. Suppose $v = 23$. Notice that

$$K_{23} = K_6 \bigcup K_{6,5} \bigcup K_{6,12} \bigcup K_{17}$$

where the vertex set of K_6 is $\{0, 1, \ldots, 5\}$, the partite sets of $K_{6,5}$ are $\{0, 1, \ldots, 5\}$ and $\{6, 7, 8, 9, 10\}$, the partite sets of $K_{6,12}$ are $\{0, 1, \ldots, 5\}$ and $\{11, 12, \ldots, 22\}$, and the vertex set of K_{17} is $\{6, 7, \ldots, 22\}$. An OC_4-decomposition of $K_{6,12}$ exists by Lemma 3.1 and an OC_4-decomposition of K_{17} exists by Theorem 2.1. These decompositions along with: $\{([1,7,0,6],[1,8,2,9]), ([4,10,3,9],[4,6,5,7]), ([5,10,0,8],[5,2,1,4]), ([3,7,2,6],[3,1,5,0]), ([2,3,8,4],[2,0,1,0])\}$ form a maximal packing of K_{23} with leave $L = C_5$ where the edge set of L is $\{(0,4), (3,4), (3,5), (5,0), (0,9)\}$.

Case 8. Suppose $v \equiv 7 \pmod{16}$, $v \geq 39$. In this case, $|E(K_v)| \equiv 5$ (mod 8). Therefore a leave L with $|E(L)| \equiv 5$ would be optimal. Also, since each vertex of K_v is of even degree and each vertex of OC_4 is of even degree, if $|E(L)| = 5$ then $L = C_5$. Notice that

$$K_v = K_{v-6} \bigcup K_{v-7,6} \bigcup K_7$$

where the vertex set of K_{v-6} is $\{0, 1, \ldots, v-7\}$, the partite sets of $K_{v-7,6}$ are $\{0, 1, \ldots, v-8\}$ and $\{v-7, v-6, \ldots, v-1\}$, and the vertex set of K_7 is $\{v-7, v-6, \ldots, v-1\}$.

Case 9. Suppose $v \equiv 8 \pmod{16}$. Then, as in Case 1, each vertex of the leave must be of odd degree. The leave must therefore consist of at least $v/2$ edges. Notice that

$$K_v = K_{v-2} \bigcup K_{v-2,2} \bigcup K_2$$

where the vertex set of K_{v-2} is $\{0, 1, \ldots, v-3\}$, the partite sets of $K_{v-2,2}$ are $\{0, 1, \ldots, v-3\}$ and $\{v-2, v-1\}$, and the vertex set of K_2 is $\{v-2, v-1\}$. Now from Case 5, we see that there is a packing of K_{v-2} with leave L_1 where $E(L_1) = \{(0,1), (2,3), \ldots, (v-6, v-5), (v-4, v-3)\} \bigcup \{(v-6, v-4), (v-4, v-5), (v-5, v-3), (v-3, v-6)\}$. By Lemma 3.2, there exists a packing of $K_{v-2,2}$ with leave $L_2 = C_4 = [v-7, v-2, v-4, v-1]$. Therefore, if we take these two packings along with $([v-4, v-1, v-2, v-7], [v-4, v-5, v-3, v-6])$, then we have a maximal packing of K_v with leave L where

$$E(L) = \{(2i, 2i+1) \mid i = 0, 1, \ldots, (v-2)/2\}$$

and $|E(L)| = v/2$.

Case 10. Suppose $v \equiv 9 \pmod{16}$. In this case, $|E(K_v)| \equiv 4 \pmod{8}$. Therefore a leave L with $|E(L)| = 4$ would be optimal. Also, as in Case 2, if $|E(L)| = 4$ then $L = C_4$. Notice that

$$K_v = K_{v-8} \bigcup K_{v-9,8} \bigcup K_9$$

where the vertex set of K_{v-8} is $\{0, 1, \ldots, v-9\}$, the partite sets of $K_{v-9,8}$ are $\{0, 1, \ldots, v-10\}$ and $\{v-8, v-7, \ldots, v-1\}$, and the vertex set of K_9 is $\{v-9, v-8, \ldots, v-1\}$. There is an OC_4-decomposition of K_{v-8} by Theorem 2.1, and there is an OC_4-decomposition of $K_{v-9,8}$ by Lemma 3.1. These decompositions along with

$$\{([v-4, v-3, v-8, v-1], [v-4, v-9, v-2, v-5]), ([v-9, v-3, v-2, v-7],$$
$$[v-9, v-8, v-4, v-6]), ([v-7, v-6, v-2, v-4], [v-7, v-1, v-9, v-5]),$$
$$([v-8, v-7, v-3, v-5], [v-8, v-2, v-1, v-6])\}$$

forms a maximal packing of K_v with leave $L = C_4 = [v-6, v-5, v-1, v-3]$.

Case 11. Suppose $v \equiv 10 \pmod{16}$. As in Case 1, it is necessary for the leave L to satisfy $E(L) \geq v/2$. Notice that

$$K_v = K_{v-2} \bigcup K_{v-2,2} \bigcup K_2$$

where the vertex set of K_{v-2} is $\{0, 1, \ldots, v-3\}$, the partite sets of $K_{v-2,2}$ are $\{0, 1, \ldots, v-3\}$ and $\{v-2, v-1\}$, and the vertex set of K_2 is $\{v-2, v-1\}$. There is an OC_4-decomposition of K_{v-2} with leave L_1 where $|E(L_1)| = (v-2)/2$ by Case 8 and there is an OC_4-decomposition of $K_{v-2,2}$ by Lemma 3.1. Therefore there exists a maximal packing of K_v with leave L where $|E(L)| = |E(L_1)| + |E(K_2)| = v/2$.

Case 12. Suppose $v \equiv 11 \pmod{16}$. In this case, $|E(K_v)| \equiv 7 \pmod{8}$. Therefore a leave L with $|E(L)| = 7$ would be optimal. Notice that

$$K_v = K_{v-10} \bigcup K_{v-11,10} \bigcup K_{11}$$

where the vertex set of K_{v-10} is $\{0, 1, \ldots, v-11\}$, the partite sets of $K_{v-11,10}$ are $\{0, 1, \ldots, v-12\}$ and $\{v-10, v-9, \ldots, v-1\}$, and the vertex set of K_{11} is $\{v-11, v-10, \ldots, v-1\}$. There is an OC_4-decomposition of K_{v-10} by Theorem 2.1 and there is an OC_4-decomposition of $K_{v-11,10}$ Lemma 3.1. Take these decompositions along with:

$$\{([v-6, v-9, v-4, v-11], [v-6, v-3, v-7, v-8]),$$

$$([v-5,v-10,v-7,v-6],[v-5,v-11,v-3,v-8]),$$
$$([v-4,v-5,v-9,v-8],[v-4,v-7,v-11,v-10]),$$
$$([v-9,v-11,v-8,v-10],[v-9,v-2,v-7,v-1]),$$
$$([v-6,v-4,v-3,v-10],[v-6,v-1,v-11,v-2]),$$
$$([v-5,v-3,v-9,v-7],[v-5,v-1,v-4,v-2])\}.$$

This gives a maximal packing of K_v with leave L where

$$E(L)=\{(v-2,v-10),(v-2,v-8),(v-2,v-3),(v-1,v-10),$$
$$(v-1,v-8),(v-1,v-3),(v-2,v-1)\}$$

and $|E(L)|=7$.

Case 13. Suppose $v \equiv 12 \pmod{16}$. Then as in Case 1, each vertex of the leave L must be of odd degree and so $|E(L)| \geq v/2$. Now $|E(K_v)| \equiv 2 \pmod 8$ and $v/2 \equiv 6 \pmod 8$, so a maximal packing will have a leave L where $|E(L)| \geq v/2+4$. Notice that

$$K_v = K_{v-10} \bigcup K_{v-10,10} \bigcup K_{10}$$

where the vertex set of K_{v-10} is $\{0,1,\ldots,v-11\}$, the partite sets of $K_{v-10,10}$ are $\{0,1,\ldots,v-11\}$ and $\{v-10,v-9,\ldots,v-1\}$, and the vertex set of K_{10} is $\{v-10,v-9,\ldots,v-1\}$. Now from Case 2, we see that there is a packing of K_{v-10} with leave L_1 where $|E(L_1)| = (v-10)/2$. By Case 10, there is a packing of K_{10} with leave L_2 where $|E(L_2)|=5$. By Lemma 3.2, there is a packing of $K_{v-10,10}$ with leave L_3 where $|E(L_3)|=4$. Therefore there exists a maximal packing of K_v with leave L where

$$|E(L)| = |E(L_1)|+|E(L_2)|+|E(L_3)| = v/2+4.$$

Case 14. Suppose $v \equiv 13 \pmod{16}$. Then $|E(K_v)| \equiv 6 \pmod 8$. Therefore a leave L with $|E(L)|=6$ would be optimal. Notice that

$$K_v = K_{v-12} \bigcup K_{v-13,12} \bigcup K_{13}$$

where the vertex set of K_{v-12} is $\{0,1,\ldots,v-13\}$, the partite sets of $K_{v-13,12}$ are $\{0,1,\ldots,v-14\}$ and $\{v-12,v-11,\ldots,v-1\}$, and the vertex set of K_{12} is $\{v-13,v-12,\ldots,v-1\}$. There is an OC_4-decomposition of K_{v-12} by Theorem 2.1. There is an OC_4-decomposition of $K_{v-13,12}$ by Lemma 3.1. Now for K_{13} notice that

$$K_{13} = K_9 \bigcup K_{8,4} \bigcup K_4 \bigcup S_4$$

where the vertex set of K_9 is $\{v-13, v-12, \ldots, v-5\}$, the partite sets of $K_{8,4}$ are $\{v-13, v-12, \ldots, v-6\}$ and $\{v-4, v-3, v-2, v-1\}$, the vertex set of K_4 is $\{v-4, v-3, v-2, v-1\}$, the vertex set of S_4 is $\{v-5, v-4, v-3, v-2, v-1\}$ and the edge set of S_4 is $\{(v-5, v-4), (v-5, v-3), (v-5, v-2), (v-5, v-1)\}$. By Case 9, K_9 can be packed with OC_4's and a leave of $C_4 = [v-5, v-6, v-7, v-8]$. By Lemma 3.1, there exists an OC_4-decomposition of $K_{8,4}$. If we take the edges of these packings and decompositions along with $\{([v-5, v-6, v-7, v-8], [v-5, v-4, v-3, v-2])\}$ then we have a maximal packing of K_v with leave L where

$$|E(L)| = |\{(v-5, v-3), (v-3, v-1), (v-1, v-5), (v-1, v-4), (v-4, v-2), (v-2, v-1)\}| = 6.$$

In fact, in this construction $L = OC_3$.

Case 15. Next, suppose $v \equiv 14 \pmod{16}$. Then as in Case 1, each vertex of the leave must be of odd degree and $|E(L)| \geq v/2$. Now $|E(K_v)| \equiv 3 \pmod 8$ and $v/2 \equiv 7 \pmod 8$, so a maximal packing will have a leave L where $|E(L)| \geq v/2 + 4$. First, if $v = 14$ then consider

$$\{([1,10,0,9],[1,11,2,12]),([3,9,2,10],[3,11,14,13]), ([5,10,4,9],[5,12,6,13]),([7,11,6,9],[7,12,8,13]), ([11,9,12,0],[11,13,10,8]),([13,12,10,9],[13,0,2,1]), ([5,2,7,0],[5,8,4,3]),([8,2,4,6],[8,7,5,1]), ([6,1,4,5],[6,0,8,3]),([7,6,2,3],[7,4,0,1])\}.$$

This is a maximal packing of K_{14} with OC_4's and leave L with

$$E(L) = \{(0,3),(1,3),(2,13),(3,12),(4,12),(5,11), (6,10),(7,10),(8,9),(10,11),(11,12)\}$$

and $|E(L)| = 11$. Next, suppose $v \equiv 14 \pmod{16}$, $v \geq 30$. Notice that

$$K_v = K_{v-22} \bigcup K_{v-22,22} \bigcup K_{22}$$

where the vertex set of K_{v-22} is $\{0, 1, \ldots, v-23\}$, the partite sets of $K_{v-22,22}$ are $\{0, 1, \ldots, v-23\}$ and $\{v-22, v-21, \ldots, v-1\}$, and the vertex set of K_{22} is $\{v-22, v-21, \ldots, v-1\}$. By Case 8, there is a packing of K_{v-22} with leave L_1 where $|E(L_1)| = (v-22)/2$. By Lemma 3.1, there is an OC_4-decomposition of $K_{v-22,22}$. By Case 5, there is a packing of K_{22} with leave L_2 where $|E(L_2)| = 15$. Therefore there is a maximal packing of K_v with leave L where

$$|E(L)| = |E(L_1)| + |E(L_2)| = v/2 + 4.$$

Case 16. Suppose $v \equiv 15 \pmod{16}$. Then $|E(K_v)| \equiv 1 \pmod 8$. As in Case 2, each vertex of the leave L of a maximal packing must be of even degree. Clearly, this cannot happen with $|E(L)| = 1$, and so it is necessary that $|E(L)| \geq 9$. Notice that

$$K_v = K_{v-14} \bigcup K_{v-15,14} \bigcup K_{15}$$

where the vertex set of K_{v-14} is $\{0, 1, \ldots, v-15\}$, the partite sets of $K_{v-15,14}$ are $\{0, 1, \ldots, v-16\}$ and $\{v-14, v-13, \ldots, v-1\}$, and the vertex set of K_{15} is $\{v-15, v-14, \ldots, v-1\}$. There is an OC_4-decomposition of K_{v-14} by Theorem 2.1. There is a decomposition of $K_{v-15,14}$ by Lemma 3.1. Next, notice that

$$K_{15} = K_9 \bigcup K_{8,6} \bigcup K_7$$

where the vertex set of K_9 is $\{v-15, v-14, \ldots, v-7\}$, the partite sets of $K_{8,6}$ are $\{v-15, v-14, \ldots, v-8\}$ and $\{v-6, v-5, \ldots, v-1\}$, and the vertex set of K_7 is $\{v-7, v-6, \ldots, v-1\}$. By Case 9, there is a packing of K_9 with leave $L_1 = C_4 = [v-7, v-8, v-9, v-10]$. By Lemma 3.1, there is an OC_4-decomposition of $K_{8,6}$. If we take the edges of these packings and decompositions along with $\{([v-7, v-8, v-9, v-10], [v-7, v-6, v-4, v-2]), ([v-3, v-7, v-1, v-2], [v-3, v-4, v-5, v-6])\}$, then we have a maximal packing of K_v with leave L where

$$E(L) = \{(v-1,v-3),(v-1,v-4),(v-1,v-5),(v-1,v-6),(v-2,v-5),$$
$$(v-2, v-6), (v-3, v-5), (v-4, v-7), (v-5, v-7)\}$$

and $|E(L)| = 9$.

Case 17. Suppose $v \equiv 0 \pmod{16}$, $v > 16$. Then as in Case 1, each vertex of the leave must be of odd degree and $|E(L)| \geq v/2$. Notice that

$$K_v = K_{v-2} \bigcup K_{v-2,2} \bigcup K_2$$

where the vertex set of K_{v-2} is $\{0, 1, \ldots, v-3\}$, the partite sets of $K_{v-2,2}$ are $\{0, 1, \ldots, v-3\}$ and $\{v-2, v-1\}$, and the vertex set of K_2 is $\{v-2, v-1\}$. By Case 14, there exists a packing of K_{v-2} with leave L_1 where $E(L_1) = \{(2i, 1+2i) \mid i = 0, 1, \ldots, (v-4)/2\} \bigcup \{(v-3, v-5), (v-5, v-4), (v-4, v-6), (v-6, v-3)\}$. By Lemma 3.2, there exists a packing of $K_{v-2,2}$ with leave $L_2 = C_4$ where $E(L_2) = \{(v-3, v-2), (v-2, v-7), (v-7, v-1), (v-1, v-3)\}$. If we take the edges of these packings and decompositions along with $\{([v-3, v-5, v-4, v-6], [v-3, v-2, v-7, v-1])\}$ then we have a maximal packing of K_v with leave L where $E(L) = \{(2i, 1+2i) \mid i = 0, 1, \ldots, (v-2)/2\}$ and $|E(L)| = v/2$. ∎

4 Conclusion

We have given necessary and sufficient conditions for the existence of an OC_4-decomposition and an OC_4-packing of K_v. In summary: An OC_4-decomposition of K_v exists if and only if $v \equiv 1 \pmod{16}$, and a maximal packing of K_v with OC_4's and leave L satisfies the following:

1. if $v \equiv 0$ or $2 \pmod 8$, then $|E(L)| = v/2$,
2. if $v \equiv 4$ or $6 \pmod 8$, then $|E(L)| = v/2 + 4$,
3. if $v \equiv 1, 3, 7, 9, 11$ or $13 \pmod{16}$, $v \neq 7$, then $|E(L)| = |E(K_v)| \pmod 8$,
4. if $v = 7$, then $|E(L)| = 13$,
5. if $v \equiv 5$ or $15 \pmod{16}$, then $|E(L)| = |E(K_v)| \pmod 8 + 8$.

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