Bicyclic, Rotational, and Reverse Mixed Triple Systems

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Abstract

A *mixed triple system* is a decomposition of the complete mixed graph into one of the partial orientations of a 3-cycle which consists of two arcs and one edge. An automorphism of a mixed triple system which consists of two disjoint cycles is said to be *bicyclic*. An automorphism consisting of a fixed point and a single cycle is *rotational*. An automorphism which, when applied to the vertex set of a mixed triple system listed in a particular order, reverses the order of the vertices, is said to be *reverse*. Necessary and sufficient conditions are given for the existence of bicyclic, rotational, and reverse mixed triple systems for each of the three types of mixed triple systems.

1 Introduction

Let K_v denote the complete graph on v vertices. For graph g, a *g-decomposition* of K_v is a set $\gamma = \{g_1, g_2, \dots, g_n\}$ of edge disjoint subgraphs of K_v each of which is isomorphic to g and $\bigcup_{i=1}^{n} E(g_i) = E(K_v)$, where $E(G)$ is the edge set of graph G. A decomposition of the complete directed graph, D_v, is similarly defined in terms of arcs and arc sets. Throughout this paper, we

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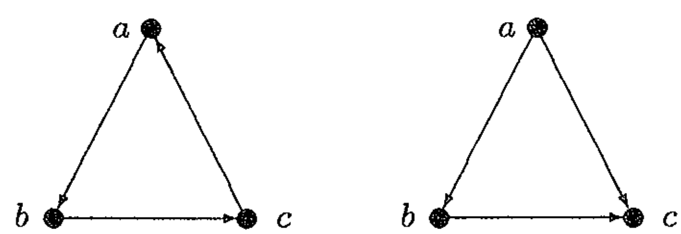
denote the edge between vertex v and vertex w as the unordered pair (v, w). We denote the arc from vertex v to vertex w as the ordered pair $[v, w]$.

An *automorphism* of a g-decomposition of K_v is a permutation of the vertex set of K_v which fixes the set γ (with automorphism of a directed graph decomposition similarly defined). An automorphism consisting of a single cycle is *cyclic*. An automorphism which consists of two disjoint cycles is *bicyclic*. An automorphism consisting of a fixed point and a single cycle is *rotational*. An automorphism of a g-decomposition of K_v is *reverse* if, when applied to the vertices of K_v written in a particular order, that order is reversed. So a reverse automorphism consists of $v/2$ transpositions when v is even, or $(v-1)/2$ transpositions and a fixed point when v is odd.

A graph (directed graph) decomposition into isomorphic copies of a graph (directed graph) on three vertices is equivalent to a triple system. A K_3-decomposition of K_v is a *Steiner triple system* of order v, which is widely known to exist if and only if $v \equiv 1$ or 3 (mod 6). We denote the following directed graphs as d_m and d_t:



d_m d_t

A d_m-decomposition of D_v is a *Mendelsohn triple system* of order v and exists if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [11]. A d_t-decomposition of D_v is a *directed triple system* of order v and exists if and only if $v \equiv 0$ or 1 (mod 3) [10].

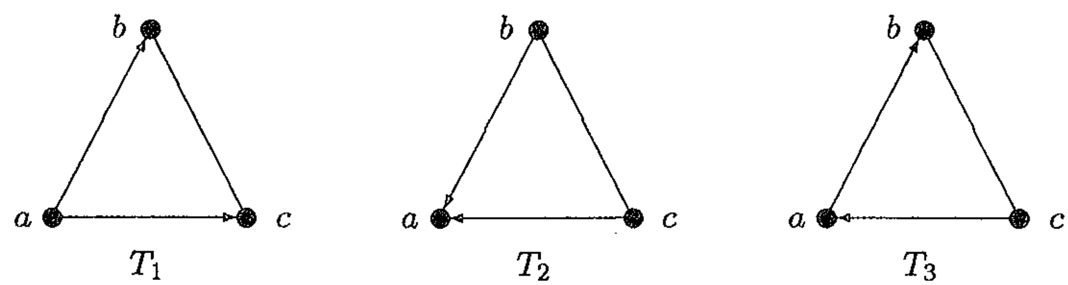
A cyclic Steiner triple system of order v exists if and only if $v \equiv 1$ or 3 (mod 6), $v \neq 9$ [12]. A cyclic Mendelsohn triple system of order v exists if and only if $v \equiv 1$ or 3 (mod 6) [5]. A cyclic directed triple system of order v exists if and only if $v \equiv 1, 4,$ or 7 (mod 12) [6].

A bicyclic Steiner triple system of order v admitting an automorphism consisting of disjoint cycles of lengths $N_1 > 1$ and N_2 where $N_1 < N_2$ exists if and only if $N_1 \equiv 1$ or 3 (mod 6), $N_1 \neq 9$, $N_1 \mid N_2$, and $v = N_1 + N_2 \equiv 1$ or 3 (mod 6) [1]. A bicyclic directed triple system of order v admitting an automorphism consisting of disjoint cycles of lengths N_1 and N_2, where $N_1 < N_2$, exists if and only if $N_1 \equiv 1, 4$, or 7 (mod 12) and $N_2 = kN_1$ where $k \equiv 2 \pmod 3$ [8].

A rotational Steiner triple system of order v exists if and only if $v \equiv 3$ or 9 (mod 24) [13]. A rotational Mendelsohn triple system of order v exists if and only if $v \equiv 1, 3$, or 4 (mod 6), $v \neq 10$ [3]. A rotational directed triple system of order v exists if and only if $v \equiv 0 \pmod 3$ [4].

A reverse Steiner triple system exists if and only if $v \equiv 1$, 3, 9, or 19 (mod 24) [7, 14, 15, 16]. A reverse directed triple system of order v exists if and only if $v \equiv 0$, 1, 3, 4, 7, or 9 (mod 12) [2]. To the authors' knowledge, neither bicyclic nor reverse Mendelsohn triple systems have been studied.

The *complete mixed graph* on v vertices, denoted M_v, is a vertex set V of cardinality v, together with a set C of ordered and unordered pairs of V such that for all $x, y \in V$, $x \neq y$, we have $(x, y), [x, y], [y, x] \in C$. There are three partial orientations of a 3-cycle which, like M_v, contain twice as many arcs as edges:



We denote T_i by the ordered triple $[a, b, c]_i$. Decompositions of M_v are defined similarly to decompositions of K_v and D_v. A T_i-decomposition of M_v is a T_i *triple system* of order v. A T_i-triple system of order v exists if and only if $v \equiv 1 \pmod 2$, except $v \in \{3, 5\}$ when $i = 3$ [9]. In fact, the constructions used in [9] take advantage of cyclic automorphisms and hence shows that the existence condition is also the condition for cyclic mixed triple systems.

The purpose of this paper is to give necessary and sufficient conditions for the existence of bicyclic, rotational, and reverse mixed triple systems.

2 Bicyclic Mixed Triple Systems

In this section, we consider bicyclic mixed triple systems of order v where $v = N_1 + N_2$ and the automorphism is $(0_1, 1_1, 2_1, \cdots, (N_1-1)_1)$ $(0_2, 1_2, 2_2, \cdots, (N_2-1)_2)$.

Lemma 2.1 *If a bicyclic mixed triple system exists with N_1 and N_2 as described above, then $N_1 \equiv 1 \pmod 2$ and $N_2 = kN_1$ for some even $k \in \mathbb{N}$.*

Proof. First, consider a mixed triple system admitting automorphism α. Suppose α fixes vertices a and b: $\alpha(a) = a$ and $\alpha(b) = b$. Now edge (a, b) is in some mixed triple, say T_i^{ab} where the vertex set of T_i^{ab} is $\{a, b, c\}$. Now if we apply α to T_i^{ab} we see that edge (a, b) is an edge of $\alpha(T_i^{ab})$. However, since edge (a, b) occurs in only one triple, it must be that $\alpha(T_i^{ab}) = T_i^{ab}$. From this observation, follows the fact that the fixed points of α form a subsystem of the original system (that is, for any a and b fixed by α, there is a triple fixed under α which contains edge (a, b), a triple fixed under α which contains arc $[a, b]$, and a triple fixed under α which contains arc $[b, a]$). Hence the number of fixed points of an automorphism must be odd. Now if π is the bicyclic automorphism, then π^{N_1} has N_1 fixed points and so $N_1 \equiv 1 \pmod 2$. Since $v = N_1 + N_2 \equiv 1 \pmod 2$, N_2 is even.

Consider π^{N_2}. It fixes points $\{0_2, 1_2, \dots, (N_2-1)_2\}$. However, $N_2 \equiv 0 \pmod 2$, so these cannot be the only fixed points. Therefore π^{N_2} must fix all v points and N_2 is a multiple of N_1. ∎

Theorem 2.1 *A bicyclic T_1-triple system exists admitting an automorphism consisting of a cycle of length N_1 and a cycle of length N_2, where $N_1 < N_2$, if and only if $N_1 \equiv 1 \pmod 2$ and $N_2 = kN_1$ for some even $k \in \mathbb{N}$.*

Proof. The necessary conditions follow from Lemma 2.1. For sufficiency, we consider cases.

Case 1. If $N_2 \equiv 0 \pmod 4$, then consider the blocks:

$$\left[0_2, \left(\frac{N_2}{4} - 1 - i\right)_2, \left(\frac{N_2}{4} + 1 + i\right)_2\right]_1 \text{ for } i = 0, 1, \dots, \frac{N_2 - 2N_1 - 6}{4},$$

$$\left[0_2, \left(\frac{3N_2}{4} - 1 - i\right)_2, \left(\frac{3N_2}{4} + i\right)_2\right]_1 \text{ for } i = 0, 1, \dots, \frac{N_2}{4} - 1,$$

$$\left[0_1, \left(\frac{N_1 - 3}{2} - i\right)_2, \left(\frac{N_2 - N_1 - 1}{2} + i\right)_2\right]_1 \text{ for } i = 0, 1, \dots, \frac{N_1 - 3}{2}.$$

If $N_1 \equiv 1 \pmod 4$, then also take the blocks:

$[0_2, i_1, (2i+1)_2]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 5}{4}$,

$\left[0_2, \left(\dfrac{N_1 - 1}{4} + i\right)_1, \left(\dfrac{N_2 - N_1 + 1}{2} + 2i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 5}{4}$,

$\left[0_2, \left(\dfrac{N_1 + 1}{2} + i\right)_1, (2 + 2i)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 5}{4}$,

$\left[0_2, \left(\dfrac{3N_1 + 1}{4} + i\right)_1, \left(\dfrac{N_2 - N_1 + 3}{2} + 2i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 5}{4}$,

$\left[0_2, \left(\dfrac{N_1 - 1}{2}\right)_1, \left(\dfrac{N_2}{4}\right)_2\right]_1$ and

$\left[0_1, (N_1 - 1)_2, \left(\dfrac{N_2 + 2N_1 - 2}{2}\right)_2\right]_1$.

If $N_1 \equiv 3 \pmod 4$, then also take the blocks:

$[0_2, i_1, (1+2i)_2]_1$ for $= 0, 1, \ldots, \dfrac{N_1 - 3}{4}$,

$\left[0_2, \left(\dfrac{N_1 + 1}{4} + i\right)_1, \left(\dfrac{N_2 - N_1 + 3}{2} + 2i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 7}{4}$,

$\left[0_2, \left(\dfrac{N_1 + 1}{2} + i\right)_1, (2 + 2i)_2\right]_1$ for $= 0, 1, \ldots, \dfrac{N_1 - 7}{4}$,

$\left[0_2, \left(\dfrac{3N_1 + 3}{4} + i\right)_1, \left(\dfrac{N_2 - N_1 + 5}{2} + 2i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 7}{4}$,

$\left[0_2, \left(\dfrac{N_1 - 1}{2}\right)_1, \left(\dfrac{N_2}{4}\right)_2\right]_1$,

$\left[0_2, \left(\dfrac{3N_1 - 1}{4}\right)_1, \left(\dfrac{N_2 - N_1 + 1}{2}\right)_2\right]_1$, and

$\left[0_1, (N_1 - 1)_2, \left(\dfrac{N_2 + 2N_1 - 2}{2}\right)_2\right]_1$.

Case 2. If $N_2 \equiv 2 \pmod 4$, then consider the blocks:

$[0_2, i_1, (2i+1)_2]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 3}{2}$,

$\left[0_2, \left(\dfrac{N_1 + 1}{2} + i\right)_1, (2 + 2i)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 3}{2}$,

$\left[0_2, \left(\dfrac{N_2 + 2N_1 - 4}{4} - i\right)_2, \left(\dfrac{N_2 + 2N_1 + 4}{4} + i\right)_2\right]_1$
for $i = 0, 1, \ldots, \frac{N_2 - 2N_1 - 4}{4}$,

$\left[0_2, \left(\dfrac{3N_2 - 2}{4} - i\right)_2, \left(\dfrac{3N_2 + 2}{4} + i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_2 - 6}{4}$,

$\left[0_1, i_2, \left(\dfrac{N_2}{2} - 1 - i\right)_2\right]_1$ for $i = 0, 1, \ldots, \dfrac{N_1 - 3}{2}$,

$\left[0_1, \left(\dfrac{N_1 - 1}{2}\right)_2, \left(\dfrac{N_2 + N_1 - 1}{2}\right)_2\right]_1$, and

$$\left[0_2, \left(\frac{N_1-1}{2}\right)_1, \left(\frac{N_2+2N_1}{4}\right)_2\right]_1.$$

In each case, the collection of blocks, along with their images under the permutation $(0_1, 1_1, \cdots, (N_1-1)_1)$ $(0_2, 1_2, \cdots, (N_2-1)_2)$, form a bicyclic T_1-triple system. ∎

Corollary 2.1 *A bicyclic T_2-triple system exists admitting an automorphism consisting of a cycle of length N_1 and a cycle of length N_2, where $N_1 < N_2$, if and only if $N_1 \equiv 1 \pmod 2$ and $N_2 = kN_1$ for some even $k \in \mathbb{N}$.*

Proof. If we reverse each of the arcs of M_v, then we get back M_v. If we reverse each of the arcs of T_1, then we get T_2 (we say that "T_1 is the *converse* of T_2"). Therefore the existence of a bicyclic T_1-triple system is equivalent to the existence of a bicyclic T_2-triple system. The result then follows from Theorem 2.1. ∎

Theorem 2.2 *A bicyclic T_3-triple system does not exist.*

Proof. Suppose, to the contrary, that such a system does exist. By Lemma 2.1, N_2 is even. The edge $(0_2, (N_2/2)_2)$ must be in some T_3, either $T_3^a = [a_2, 0_2, (N_2/2)_2]_3$ or $T_3^b = [b_1, 0_2, (N_2/2)_2]_3$. Applying $\pi^{N_2/2}$ we fix edge $(0_2, (N_2/2)$ and get $\pi^{N_2/2}(T_3^a) = [(a+N_2/2)_2, (N_2/2)_2, 0_2)_3 = T_3^{a\prime}$ and $\pi^{N_2/2}(T_3^b) = [b_1, (N_2/2)_2, 0_2]_3 = T_3^{b\prime}$. So we need $T_3^a = T_3^{a\prime}$ or $T_3^b = T_3^{b\prime}$, both contradictions. ∎

3 Rotational Mixed Triple Systems

A T_i-triple system of order v is said to be *rotational* if it admits an automorphism consisting of a fixed point and a cycle of length $v-1$. By taking $N_1 = 1$ in the previous section, we have necessary and sufficient conditions for the existence of a rotational T_i-triple system.

Theorem 3.1 *A rotational T_i-triple system of order v*

(i) *exists if and only if $v \equiv 1 \pmod 2$ when $i \in \{1, 2\}$, and*

(ii) *does not exist when $i = 3$.*

4 Reverse Mixed Triple Systems

A T_i-triple system of order v is said to be *reverse* if it admits an autmorphism consisting of a fixed point and $(v-1)/2$ transpositions. The existence of reverse T_i-triple systems follows easily from the existence of rotational T_i-triple systems.

Theorem 4.1 *A reverse T_i-triple system of order v:*

(i) *exists if and only if $v \equiv 1 \pmod 2$ when $i \in \{1, 2\}$, and*

(ii) *does not exist when $i = 3$.*

Proof. When $i \in \{1, 2\}$, for all $v \equiv 1 \pmod 2$, there exists a rotational T_i-triple system of order v admitting an automorphism π consisting of a fixed point and a cycle of length $v-1$. By considering $\pi^{(v-1)/2}$, we see that a reverse T_i-triple system exists for all $v \equiv 1 \pmod 2$.

When $i = 3$, we suppose a reverse T_3-triple system exists admitting the automorphism $\pi = (\infty)(0_1, 1_1)(0_2, 1_2)\cdots(0_{(v-1)/2}, 1_{(v-1)/2})$. The edge $(0_1, 1_1)$ must be in some triple, say $T_3^1 = [x, 0_1, 1_1]_3$. Now $\pi(T_3^1) = [\pi(x), 1_1, 0_1]_3$ contains edge $(0_1, 1_1)$ and so must also contain arcs $[x, 0_1]$ and $[1_1, x]$. However, $\pi(T_3^1)$ does not contain these arcs and this contradiction shows that no such T_3-triple system exists. ∎

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