Maximal Cyclic 4-Cycle Packings and Minimal Cyclic 4-Cycle Coverings of the Complete Graph

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Abstract. In this paper, we define an automorphism of a graph packing and of a graph covering. We consider automorphisms which consist of a single cycle (called *cyclic*) and give necessary and sufficient conditions for maximal cyclic 4-cycle packings and minimal cyclic 4-cycle coverings of the complete graph.

1 Introduction

A *g-decomposition* of (simple) graph G is a set $\gamma = \{g_1, g_2, \ldots, g_n\}$ of isomorphic copies of graph g such that $V(g_i) \subset V(G)$ for $i = 1, 2, \ldots, n$, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{n} g_i = G$. That is, a g-decomposition of G is a partitioning of $E(G)$ into the edge sets $E(g_1), E(g_2), \ldots, E(g_n)$. A large number of graph decompositions have been studied, in particular cycle decompositions of the complete graph (see, for example, [11]). For example, a C_3-decomposition of K_v is equivalent to a Steiner triple system of order v. For our purposes, we mention that it is well known that a C_4-decomposition of K_v exists if and only if $v \equiv 1 \pmod 8$.

An *automorphism* of a g-decomposition of G is a permutation, π, of $V(G)$ which fixes the set γ. If π consists of a single cycle of length $|V(G)|$, then the decomposition is said to be *cyclic*. A cyclic Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$ [12]. Many other graph decompositions and types of permutations have been studied (the literature on Steiner triple systems includes reverse automorphisms [14, 18], k-rotational automorphisms [5, 13], and bicyclic automorphisms [4]).

When a g-decomposition of G does not exist, we can ask the question "How close to a g-decomposition can we get?" There are two approaches to this question: packings and coverings. (The astute observer will notice a parallel between packings and coverings of graphs and the concept from analysis of inner and outer measure of a set, respectively.) A *g-packing* of G is a set $\gamma = \{g_1, g_2, \ldots, g_n\}$ of isomorphic copies of g such that $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, and $\cup_{i=1}^{n} g_i \subset G$. We define the *leave* of a packing as $L = G \setminus \cup_{i=1}^{n} g_i$ (that is, $E(L) = E(G \setminus \cup_{i=1}^{n} g_i)$ and $V(L)$ is the set of vertices induced by $E(L)$). A number of graphs have been studied in connection with the problem of finding maximal packings (with

minimal leaves). Maximal C_3-packings of K_v were explored by Schönheim and Spencer [15, 17]. A C_4-packing of K_v with minimal leave L exists if and only if [16]
(1) if $v \equiv 0 \pmod 2$ then $|E(L)| = v/2$,
(2) if $v \equiv 1 \pmod 8$ then $|E(L)| = 0$,
(3) if $v \equiv 3 \pmod 8$ then $|E(L)| = 3$,
(4) if $v \equiv 5 \pmod 8$ then $|E(L)| = 6$, and
(5) if $v \equiv 7 \pmod 8$ then $|E(L)| = 5$.
K_4-packings of K_v are studied in [1] and C_6-packings of K_v in [8, 9]. Some packings of noncomplete graphs have also been studied, for example some cycle packings of $K_v - K_u$ are studied in [2, 3]. An *automorphism* of a g-packing of G is a permutation, π, of $V(G)$ which fixes γ. Notice that π must also fix the leave L. If π consists of a single cycle of length $|V(G)|$, then the packing is *cyclic.* A cyclic packing is *maximal* if $|E(L)|$ is minimal. In the next section we explore maximal cyclic C_4-packings of K_v.

A *g-covering* of (simple) graph G is a set $\gamma = \{g_1, g_2, \ldots, g_n\}$ of isomorphic copies of g such that $V(g_i) \subset V(G)$ for $i = 1, 2, \ldots, n$ and $G \subset \cup_{i=1}^{n} g_i$. We wish to treat $\cup_{i=1}^{n} g_i$ as a multigraph and $E(\cup_{i=1}^{n} g_i)$ as a multiset. With this convention, we define the *padding* of a covering as the (possibly multigraph) $P = \cup_{i=1}^{n} g_i \setminus G$ (to be explicit, as with the leave of a padding, $E(P) = E(\cup_{i=1}^{n} g_i \setminus G)$ and $V(P)$ is the set of vertices induced by $E(P)$). Some graphs have been studied in connection with the problem of finding minimal coverings (with a minimal paddings). Minimal C_3-coverings of K_v were explored by Fort and Hedlund [6]. A C_4-covering of K_v with minimal padding P exists if and only if [16]
(1) if $v \equiv 0 \pmod 4$ then $|E(P)| = v/2$,
(2) if $v \equiv 2 \pmod 4$ then $|E(P)| = v/2 + 2$,
(3) if $v \equiv 1 \pmod 8$ then $|E(P)| = 0$,
(4) if $v \equiv 3 \pmod 8$ then $|E(P)| = 5$,
(5) if $v \equiv 5 \pmod 8$ then $|E(P)| = 2$, and
(6) if $v \equiv 7 \pmod 8$ then $|E(P)| = 3$.
C_6-coverings of K_v are studied in [10]. Coverings have not been as extensively studied as packings. To the author's knowledge, the only covering result which does not involve the complete graph concerns C_4-coverings of $K_v - K_u$ [7]. An *automorphism* of a g-covering of G is a permutation, π, of $V(G)$ which fixes γ. Notice that π must also fix the padding P. If π consists of a single cycle of length $|V(G)|$, then the covering is cyclic. A cyclic covering is *minimal* if $|E(P)|$ is minimal. In the final section of this paper, we give necessary and sufficient conditions for the existence of a minimal cyclic C_4-covering of K_v.

The *orbit* of some subgraph g of G under permutation π is $\{\pi^i(g) \mid i = 0, 1, \ldots, |V(G)| - 1\}$. the *length* of the orbit of g is the cardinality of the orbit of g. Throughout this paper, we assume $V(K_v) = \{0, 1, \ldots, v-1\}$

and that the cyclic permutation is $(0, 1, \cdots, v-1)$ (and hence all vertex labels should be reduced modulo v).

Lemma 1.1 *Let γ be a cyclic C_4-packing of K_v where v is odd. Then the leave L of this packing satisfies $|E(L)| \equiv 0 \pmod v$.*
Proof. If v is odd, then the length of the orbit of any edge of K_v under a cyclic automorphism is v. Since the orbits of the edges of K_v partition the set $E(K_v)$, then $|E(L)| \equiv 0 \pmod v$. ∎

Lemma 1.2 *Let γ be a cyclic C_4-covering of K_v where v is odd. Then the padding P of this covering satisfies $|E(P)| \equiv 0 \pmod v$.*
Proof. As in the proof of Lemma 1.1, the length of the orbit of any edge of K_v is v, the orbits of the edges partition the multiset $E(K_v) \cup E(P)$, and so $|E(P)| \equiv 0 \pmod v$. ∎

2 Maximal Cyclic C_4-Packings of K_v

A set of 4-cycles, $\beta = \{g_1, g_2, \ldots, g_m\}$, is a set of *base blocks* for a cyclic C_4-decomposition (or packing or covering) if the orbits of the elements of β partition γ. To prove the existence of cyclic C_4-packings (and coverings) of K_v, we will present sets of base blocks.

Theorem 2.1 *A maximal cyclic C_4-packing of K_v satisfies:*
(1) *if $v \equiv 0 \pmod 2$, then $|E(L)| = v/2$,*
(2) *if $v \equiv 1 \pmod 8$, then $|E(L)| = 0$,*
(3) *if $v \equiv 3 \pmod 8$, then $|E(L)| = v$,*
(4) *if $v \equiv 5 \pmod 8$, then $|E(L)| = 2v$, and*
(5) *if $v \equiv 7 \pmod 8$, then $|E(L)| = 3v$.*

Proof. We consider several cases.

Suppose $v \equiv 0 \pmod 4$. Then each vertex of K_v is of odd degree. Since each vertex of C_4 is of even degree, then in the leave, each vertex will be of odd degree. Therefore in a maximal packing, there must be at least $v/2$ edges in the leave. Consider the blocks $\{[0, i, v/2, v/2+i] \mid i = 1, 2, \ldots, v/4-1\} \cup \{[0, v/4, v/2, 3v/4]\}$. This is a set of base blocks for a cyclic C_4-packing of K_v with leave L satisfying $E(L) = \{(j, v/2+j) \mid j = 1, 2, \ldots, v/2\}$, so $|E(L)| = v/2$ and the packing is maximal.

Suppose $v \equiv 1 \pmod 8$. Consider the blocks $\{[0, 4i-3, 8i-3, 4i-1] \mid i = 1, 2, \ldots, (v-1)/8\}$. This is a set of base blocks for a cyclic C_4-decomposition of K_v.

Suppose $v \equiv 2 \pmod 4$. As in the case of $v \equiv 0 \pmod 8$, a maximal packing must have a leave with at least $v/2$ edges. Consider the blocks

$\{[0,i,v/2,v/2+i] \mid i=1,2,\ldots,(v-2)/4\}$. This set is a set of base blocks for a cyclic C_4-packing of K_v with leave L satisfying $E(L)=\{(j,v/2+j) \mid j=1,2,\ldots,v/2\}$, so $|E(L)|=v/2$ and the packing is maximal.

Suppose $v \equiv 3 \pmod 8$. By Lemma 1.1, we know that $|E(L)| \equiv 0 \pmod v$. Since no decomposition exists when $v \equiv 3 \pmod 8$, in a packing it is necessary that $|E(L)| \geq v$. Consider the blocks $\{[0,4i-3,8i-3,4i-1] \mid i=1,2,\ldots,(v-3)/8\}$. This set is a set of base blocks for a cyclic C_4-packing of K_v with leave L satisfying $E(L)=\{(j,(v-1)/2+j) \mid j=1,2,\ldots,v\}$, so $|E(L)|=v$ and the packing is maximal.

Suppose $v \equiv 5 \pmod 8$. By Lemma 1.1, we know that $|E(L)| \equiv 0 \pmod v$. In this case, $|E(K_v)| \equiv 2 \pmod 4$ and so $|E(L)| \equiv 2 \pmod 4$ is also necessary. Therefore a cyclic packing with $|E(L)|=2v$ would be maximal. Consider the blocks $\{[0,4i-3,8i-3,4i-1] \mid i=1,2,\ldots,(v-5)/8\}$. This set is a set of base blocks for a cyclic C_4-packing of K_v with leave L satisfying $E(L)=\{(j,(v-3)/2+j),(j,(v-1)/2+j) \mid j=1,2,\ldots,v\}$, so $|E(L)|=2v$ and the packing is maximal.

Suppose $v \equiv 7 \pmod 8$. By Lemma 1.1, we know that $|E(L)| \equiv 0 \pmod v$. In this case, $|E(K_v)| \equiv 1 \pmod 4$ and so $|E(L)| \equiv 1 \pmod 4$ is also necessary. Therefore a cyclic packing with $|E(L)|=3v$ would be maximal. Consider the blocks $\{[0,4i-3,8i-3,4i-1] \mid i=1,2,\ldots,(v-7)/8\}$. This set is a set of base blocks for a cyclic C_4-packing of K_v with leave L satisfying $E(L)=\{(j,(v-5)/2+j),(j,(v-3)/2+j),(j,(v-1)/2+j) \mid j=1,2,\ldots,v\}$, so $|E(L)|=3v$ and the packing is maximal. ∎

3 Minimal Cyclic C_4-Coverings of K_v

We now address coverings.

Theorem 3.1 *A minimal cyclic C_4-covering of K_v satisfies:*
(1) *if $v \equiv 0 \pmod 4$, then $|E(P)|=v/2$,*
(2) *if $v \equiv 1 \pmod 8$, then $|E(P)|=0$,*
(3) *if $v \equiv 2 \pmod 4$, then $|E(P)|=3v/2$,*
(4) *if $v \equiv 3 \pmod 8$, then $|E(P)|=3v$,*
(5) *if $v \equiv 5 \pmod 8$, then $|E(P)|=2v$,*
(6) *if $v \equiv 7 \pmod 8$, then $|E(P)|=v$.*

Proof. We consider several cases.

Suppose $v \equiv 0 \pmod 4$. Then each vertex of K_v is of odd degree. Since each vertex of C_4 is of even degree, then in the padding, each vertex will be of odd degree. Therefore in a minimal covering, there must be at least $v/2$ edges in the padding. Consider the blocks $\{[0,i,v/2,v/2-i] \mid i=1,2,\ldots,v/4-1\}\bigcup\{[0,v/2,v/4,3v/4]\}$. This set is a set of base blocks for a cyclic C_4-covering of K_v with padding P satisfying $E(P) = \{(j, v/2+j) \mid j = 1, 2, \ldots, v/2\}$, so $|E(P)| = v/2$ and the covering is minimal.

Suppose $v \equiv 1 \pmod 8$. Then there is a cyclic C_4-decomposition of K_v, as seen in Theorem 2.1, and so $|E(P)| = 0$.

Suppose $v \equiv 2 \pmod 4$. In this case, every block has an orbit that contains a multiple of v edges (namely, either $2v$ or $4v$). Therefore it is necessary that $|E(K_v)| + |E(P)| \equiv 0 \pmod v$. Since $|E(K_v)|$ is an odd multiple of $v/2$, then $|E(P)|$ must also be an odd multiple of $v/2$. Now $|E(K_v)| + v/2 \equiv 2 \pmod 4$, so $|E(P)| \geq 3v/2$ is necessary. Consider the blocks $\{[0, i, v/2, v/2+i] \mid i = 1, 2, \ldots, (v-2)/4\} \bigcup \{[0, 1, v/2+1, v/2]\}$. This set is a set of base blocks for a cyclic C_4-covering of K_v with padding P satisfying $E(P) = \{(j, v/2+j) \mid j = 1, 2, \ldots, v/2\} \cup \{(j, j+1) \mid j = 1, 2, \ldots, v\}$, so $|E(P)| = 3v/2$ and the covering is minimal.

Suppose $v \equiv 3 \pmod 8$. By Lemma 1.2, we know that $|E(P)| \equiv 0 \pmod v$. In this case, $|E(K_v)| \equiv 3 \pmod 4$ and so $|E(P)| \equiv 1 \pmod 4$ is also necessary. Therefore a cyclic covering with $|E(P)| \equiv 3v$ would be minimal. Consider the blocks $\{[0, 4i-3, 8i-3, 4i-1] \mid i = 1, 2, \ldots, (v+5)/8\}$. This set is a set of base blocks for a cyclic C_4-packing of K_v with padding P satisfying $E(P) = \{(j, (v-5)/2+j), (j, (v-3)/2+j), (j, (v-1)/2+j) \mid j = 1, 2, \ldots, v\}$, so $|E(P)| = 3v$ and the covering is minimal.

Suppose $v \equiv 5 \pmod 8$. By Lemma 1.2, we know that $|E(P)| \equiv 0 \pmod v$. In this case, $|E(K_v)| \equiv 2 \pmod 4$ and so $|E(P)| \equiv 2 \pmod 4$ is also necessary. Therefore a cyclic covering with $|E(P)| \equiv 2v$ would be minimal. Consider the blocks $\{[0, 4i-3, 8i-3, 4i-1] \mid i = 1, 2, \ldots, (v+3)/8\}$. This set is a set of base blocks for a cyclic C_4-packing of K_v with padding P satisfying $E(P) = \{(j, (v-3)/2+j), (j, (v-1)/2+j) \mid j = 1, 2, \ldots, v\}$, so $|E(P)| = 2v$ and the covering is minimal.

Suppose $v \equiv 7 \pmod 8$. By Lemma 1.2, we know that $|E(P)| \equiv 0 \pmod v$. Since no decomposition exists when $v \equiv 7 \pmod 8$, in a covering it is necessary that $|E(P)| \geq v$. Consider the blocks $\{[0, 4i-3, 8i-3, 4i-1] \mid i = 1, 2, \ldots, (v+1)/8\}$. This set is a set of base blocks for a cyclic C_4-covering of K_v with padding P satisfying $E(P) = \{(j, (v-1)/2+j) \mid j = 1, 2, \ldots, v\}$, so $|E(P)| = v$ and the covering is minimal. ∎

4 Conclusion

Notice that in Section 2, we studied maximal cyclic C_4-packings, as applied to the less general cyclic maximal C_4-packings. If we first require that the packings are maximal *and then* explore the constraint of having a cyclic automorphism, then Theorem 2.1 implies:

Corollary 4.1 *A cyclic maximal C_4-packing of K_v exists if and only if $v \equiv 0, 1, 2, 4,$ or $6 \ (mod\ 8)$.*

Similarly, Theorem 3.1 implies:

Corollary 4.2 *A cyclic minimal C_4-covering of K_v exists if and only if $v \equiv 0, 1,$ or $4 \ (mod\ 8)$.*

Therefore, Theorem 2.1, Theorem 3.1, Corollary 4.1, and Corollary 4.2 give us necessary and sufficient conditions for (respectively) the existence of a maximal cyclic C_4-packing, a minimal cyclic C_4-covering, a cyclic maximal C_4-packing, and a cyclic minimal C_4-covering of K_v.

References

[1] A. Brouwer, Optimal Packings of K_4's into a K_n, *Journal of Combinatorial Theory, Series A*, **26**(3) (1979), 278–297.

[2] D. Bryant, C. Rodger, and E. Spicer, Embeddings of m-cycle Systems and Incomplete m-cycle Systems: $m \leq 14$, *Discrete Mathematics*, **171** (1997), 55–75.

[3] D. Bryant and A. Khodkar, Maximum Packings of $K_v - K_u$ with Triples, *Ars Combinatoria*, **55** (2000), 259–270.

[4] R. Calahan and R. Gardner, Bicyclic Steiner Triple Systems, *Discrete Math.*, **128** (1994), 35–44.

[5] C. Colbourn and Z. Jiang, The Spectrum for Rotational Steiner Triple Systems, *Journal of Combinatorial Designs*, **4**(3) (1996), 205–217.

[6] M. Fort and G. Hedlund, Minimal Coverings of Pairs by Triples, *Pacific Journal of Mathematics*, **8** (1958), 709–719.

[7] R. Gardner, C. Nguyen, and S. Lavoie, 4-Cycle Coverings of the Complete Graph with a Hole, submitted.

[8] J. Kennedy, Maximum Packings of K_n with Hexagons, *Australasian Journal of Combinatorics*, **7** (1993), 101–110.

[9] J. Kennedy, Maximum Packings of K_n with Hexagons: Corrigendum, *Australasian Journal of Combinatorics*, **10** (1994), 293.

[10] J. Kennedy, Minimum Coverings of K_n with Hexagons, *Autstralasian Journal of Combinatorics*, **16** (1997), 295–303.

[11] C. C. Lindner and C. A. Rodger, Decompositions into Cycles II: Cycle Systems, *in* "Contemporary Design Theory," (J. H. Dinitz and D. R. Stinson, Ed.) John Wiley, New York, 1992, pp. 325-369.

[12] R. Peltesohn, Eine Losung der beiden Heffterschen Differenzenprobleme, *Compositio Math.* , **6** (1939), 251–257.

[13] K. T. Phelps and A. Rosa, Steiner Triple Systems with Rotational Automorphisms, *Discrete Math.*, **33** (1981), 57–66.

[14] A. Rosa, On Reverse Steiner Triple Systems, *Discrete Math.*, **1** (1972), 61–71.

[15] J. Schönheim, On Maximal Systems of k-Tuples, *Studia Sci. Math. Hungarica* (1966), 363–368.

[16] J. Schönheim and A. Bialostocki, Packing and Covering of the Complete Graph with 4-Cycles, *Canadian Mathematics Bulletin*, **18**(5) (1975), 703–708.

[17] J. Spencer, Maximal Consistent Families of Triples, *Journal of Combinatorial Theory*, **5** (1968), 1–8.

[18] L. Teirlinck, The Existence of Reverse Steiner Triple Systems, *Discrete Math.*, **6** (1973), 301–302.