Decompositions of the Complete Symmetric Digraph into Orientations of the 4-Cycle with a Pendant Edge

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Abstract. There are twenty orientations of the 4-cycle with a pendant edge. We give necessary and sufficient conditions for the decomposition of the complete symmetric digraph on v vertices into each of these digraphs.

1 Introduction

A *g-decomposition* of (simple) graph G is a set $\gamma = \{g_1, g_2, \ldots, g_n\}$ of isomorphic copies of graph g, called *blocks*, such that $V(g_i) \subset V(G)$ for $i = 1, 2, \ldots, n$, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{n} g_i = G$. That is, a g-decomposition of G is a partitioning of $E(G)$ into the edge sets $E(g_1), E(g_2), \ldots, E(g_n)$. The definition of a decomposition of a digraph is similarly defined, with edge sets replaced with arc sets.

Graph and digraph decompositions are a widely studied area of design theory [3]. Probably the best known graph decompositions involve decompositions of the complete graph on v vertices, K_v, into cycles of a given length. For example, a 3-cycle decomposition of K_v is equivalent to a Steiner triple system and exists if and only if $v \equiv 1$ or 3 (mod 6) [9]. It is well known that a 4-cycle decomposition of K_v exists if and only if $v \equiv 1$ (mod 8). Many other decompositions of K_v into copies of small graphs have been studied [4]. Of particular interest to us, is a decomposition of K_v into copies of $L = C_3 \cup \{e\}$, the 3-cycle with a pendant edge. These exist if and only if $v \equiv 0$ or 1 (mod 8) [2]. A decomposition of K_v into copies of $H = C_4 \cup \{e\}$, the 4-cycle with a pendant edge, exists if and only if $v \equiv 0$ or 1 (mod 5), $v \neq 10$ [1].

Two nonisomorphic digraphs are determined by putting an orientation on a 3-cycle: The 3-circuit and the transitive triple. A decomposition of the complete symmetric digraph on v vertices, D_v, into 3-circuits is equivalent to a Mendelsohn triple system of order v, and such systems exist if and only if $v \equiv 0$ or 1 (mod 3), $v \neq 6$ [8]. A decomposition of D_v into transitive triples is equivalent to a directed triple system of order v and such a system exists if and only if $v \equiv 0$ or 1 (mod 3) [7]. There are four orientations of a 4-cycle. These are given in Figure 1. A 4−circuit decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 4$ [10]. An X−decomposition of D_v exists if and only if $v \equiv 0$ or 1 (mod 4), $v \neq 5$, a Y−decomposition of D_v

exists if and only if $v \equiv 0$ or 1 (mod 4), $v \notin \{4, 5\}$, and a Z–decomposition of D_v exists if and only if $v \equiv 1 \pmod 4$ [6].

Eight digraphs are determined by putting an orientation on the graph $L = C_3 \cup \{e\}$. Necessary and sufficient conditions for the existence of a decomposition of D_v into each of the orientations of $L = C_3 \cup \{e\}$ are given in [5]. Twenty digraphs are determined by putting an orientation on the graph $H = C_4 \cup \{e\}$. Half of these digraphs are given in Figure 2 (the others are the converses of those given). We denote the digraph of Figure 2 which is labeled C_{4i} as $[a, b, c, d; e]_{C4i}$, the digraph which is labeled X_{1i} as $[a, b, c, d; e]_{X1i}$, and so forth. The purpose of this paper is to give necessary and sufficient conditions for the existence of a decomposition of D_v into each of the orientations of H.

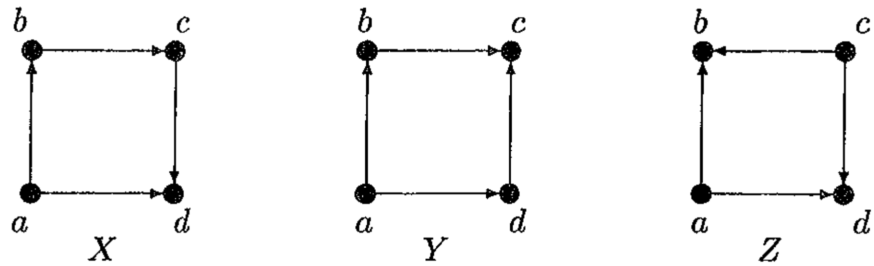

Figure 1. The four orientations of a 4-cycle are the 4-circuit and the graphs X, Y, and Z given here.

2 The Decompositions

In this section, we give necessary and sufficient conditions for the existence of a decomposition of the complete digraph into each of the twenty digraphs of Figure 2. Decompositions into the converse of these digraphs follows trivially. Since D_v has $v(v-1)$ arcs and each digraph of Figure 2 has 5 arcs, we have the following necessary condition.

Lemma 2.1 *If a decomposition of D_v into one of the digraphs of Figure 2 exists, then $v \equiv 0$ or 1 (mod 5).*

We now show that certain decompositions do not exist for some small values of v.

Lemma 2.2 *The following decompositions of D_v do not exist: an X_{2i} decomposition of D_5, an X_{2e} decomposition of D_5, a Y_{1i} decomposition of D_v for $v = 5, 6$, a Y_{1e} decomposition of D_v for $v = 5, 6$, a Z_{1i} decomposition of D_5, a Z_{1e} decomposition of D_5, a Z_{2i} decomposition of D_5, and a Z_{2e} decomposition of D_5.*

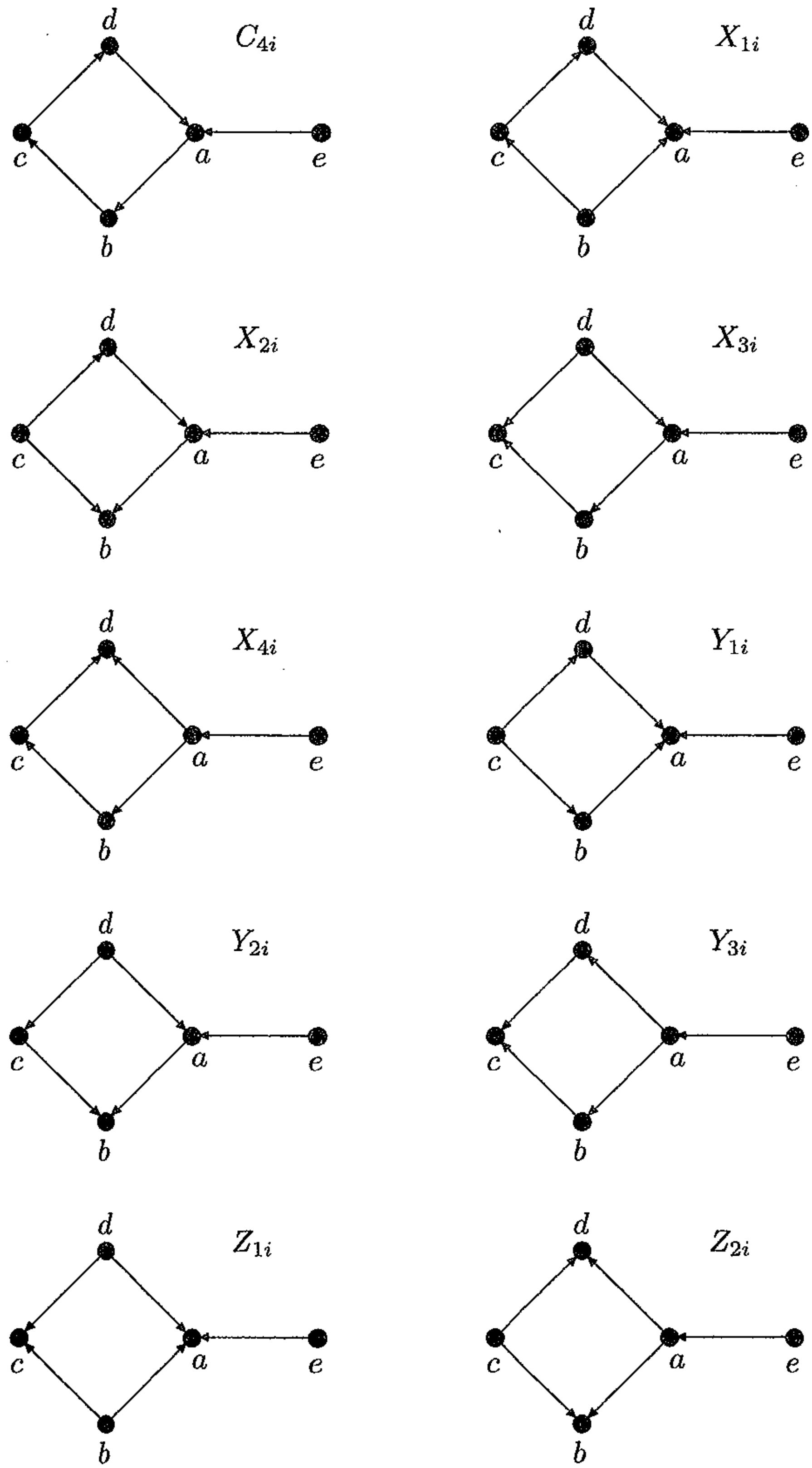

Figure 2. Here are half of the orientations of a 4-cycle with a pendant edge. The converses of these are the other ten orientations. We denote the converse of each A_{ni} as A_{ne} for $A \in \{C, X, Y, Z\}$ and $n \in \{1, 2, 3, 4\}$.

Proof. Since a decomposition of D_v into a particular given digraph is equivalent to a decomposition of D_v into the converse of the given digraph, we only establish the nonexistence of decompositions in the cases for which the pendant arc points into the oriented 4-cycle.

Case 1. An X_{2i} decomposition of D_5 does not exist. A computational argument for this result is given in the appendix.

Case 2. A Y_{1i} decomposition of D_5 does not exist. Suppose, to the contrary, that such a decomposition does in fact exist. We claim that such a decomposition cannot have two blocks of the form $B_1 = [a_1, b_1, x, d_1; e_1]_{Y1i}$ and $B_2 = [a_2, b_2, x, d_2; e_2]_{Y1i}$. For, if the decomposition does contain B_1 and B_2 where $B_1 \neq B_2$, then vertex x has out-degree 4 in digraph $B_1 \cup B_2$. The decomposition of D_5 consists of 4 copies of Y_{1i}. The remaining two blocks containing x must be of the form $B_3 = [x, b_3, c_3, d_3; e_3]_{Y1i}$ and $B_4 = [x, b_4, c_4, d_4; e_4]_{Y1i}$. Now x has out-degree 4 and in-degree 6 in digraph $B_1 \cup B_2 \cup B_3 \cup B_4$. This is a contradiction because in D_5, x has in-degree 4 and therefore the decomposition cannot include blocks of the forms B_1 and B_2.

Let $V(D_5) = \{1, 2, 3, 4, 5\}$. Without loss of generality, one block of the decomposition is $B_1 = [1, 2, 3, 4; 5]_{Y1i}$. In B_1, vertex 1 has in-degree 3 and out-degree 0. Next, in order to get the in-degree of vertex 1 up to 4, vertex 1 must be in a block B_2 either of the form $[a, 1, c, d; e]_{Y1i}$ or of the form $[a, b, c, 1; e]_{Y1i}$. But then $c \neq 1$ (1 is already a vertex of block B_2), $c \neq 2$ (since arc $[2, 1]$ is in B_1), $c \neq 3$ (by the earlier claim applied to blocks B_1 and B_2), $c \neq 4$ (since arc $[4, 1]$ is in B_1), and $c \neq 5$ (since arc $[5, 1]$ is in B_1). These contradictions imply that no such decomposition exists.

Case 3. A Y_{1i} decomposition of D_6 does not exist. Suppose, to the contrary, that such a decomposition does in fact exist. Such a decomposition cannot have two blocks of the form $B_1 = [a_1, b_1, x, d_1; e_1]_{Y1i}$ and $B_2 = [a_2, b_2, x, d_2; e_2]_{Y1i}$. Suppose it does contain blocks B_1 and B_2, where $B_1 \neq B_2$. (1) Suppose $B_3 = [a_3, x, c_3, d_3; e_3]_{Y1i}$ is a block. Then x has out-degree 5 and in-degree 1 in $B_1 \cup B_2 \cup B_3$. Then x is in 2 more blocks. The only way to get the in-degree of x up to 5 is to have x in a block of the form $B_4 = [x, b_4, c_4, d_4; e_4]_{Y1i}$. Now x has out-degree 5, in-degree 4 in $B_1 \cup B_2 \cup B_3 \cup B_4$. But x cannot be in a B_5 such that x has out-degree 0 and in-degree 1 in B_5. So no such decomposition exists. (2) Suppose $B_3 = [a_3, b_3, c_3, x; e_3]_{Y1i}$. This leads to the same contradiction as (1). (3) So the remaining 3 blocks containing x must be of the forms: $[x, b, c, d; e]_{Y1i}$, $[a, b, x, d; e]_{Y1i}$, or $[a, b, c, d; x]_{Y1i}$. But no combination of these three blocks with B_1 and

B_2 yield out-degree 5, in-degree 5 for x. Hence no such decomposition exists.

Case 4. A Z_{1i} decomposition of D_5 does not exist. Suppose, to the contrary, that such a decomposition does in fact exist. Let $[a,b,c,d;e]_{Z1i}$ be a block of the decomposition. Then vertex a is of in-degree 3 in this block and of in-degree 4 in D_5. Since no vertex of Z_{1i} is of in-degree 1, then no such decomposition exists.

Case 5. A Z_{2i} decomposition of D_5 does not exist. Suppose, to the contrary, that such a decomposition does in fact exist. Then there are 4 copies of Z_{2i} in this decomposition. In $Z_{2i} = [a,b,c,d;e]_{Z2i}$, only one vertex (vertex e) is of odd out-degree. So if $B_1 = [a_1,b_1,c_1,d_1;x]_{Z2i}$ is a block of such a decomposition then, since vertex x is of out-degree 4 in D_5, then there must be another block of $B_2 = [1_2,b_2,c_2,d_2;x]_{Z2i} \neq B_1$ in the decomposition. But then vertex x has in-degree 0 and out-degree 2 in digraph $B_1 \cup B_2$. Therefore, vertex x must be of in-degree 4 and out-degree 2 in the remaining two blocks. However, it is not possible for two copies of Z_{2i} to satisfy this condition. Hence, no such decomposition exists. ■

We now show that the necessary conditions of Lemmas 2.1 and 2.2 are sufficient.

Theorem 2.3 *A decomposition of D_v into each of the digraphs of Figure 2 exists if and only if $v \equiv 0$ or 1 (mod 5), with the following exceptions: For X_{2i} and X_{2e}, $v \neq 5$; for Y_{1i} and Y_{1e}, $v \notin \{5,6\}$; for Z_{1i} and Z_{1e}, $v \neq 5$; and for Z_{2i} and Z_{2e}, $v \neq 5$.*

Proof. The necessary conditions follow from Lemmas 2.1 and 2.2. We now establish sufficiency in several cases. In cases 1–8, reduce vertex labels modulo v when $v \equiv 1 \pmod 5$ and reduce vertex labels modulo $v-1$ when $v \equiv 0 \pmod 5$.

Case 1. For a C_{4i} decomposition of D_v for $v = 5\ell$, consider: $\{[j, 1+2i+j, 5\ell-2+j, 5\ell-3-2i+j; 3\ell-2-i+j]_{C4i} \mid i = 0,1,\ldots,\ell-2, j = 0,1,\ldots 5\ell-2\} \cup \{[j, 2\ell+j, \infty, 3\ell+j; 2\ell-1+j]_{C4i} \mid j = 0,1,\ldots,5\ell-2\}$.

For a C_{4i} decomposition of D_v for $v = 5\ell+1$, consider: $\{[j, 1+2i+j, 5\ell+j, 5\ell-1-2i+j; 2\ell+1+i+j]_{C4i} \mid i = 0,1,\ldots,\ell-1, j = 0,1,\ldots,5\ell\}$.

Case 2. For an X_{1i} decomposition of D_v for $v = 5\ell$, consider: $\{[j, \ell+i+j, \ell-1+j, 4\ell-2-i+j; 5\ell-2-i+j]_{C4i} \mid i = 0,1,\ldots,\ell-2, j =$

$0, 1, \ldots, 5\ell - 2\} \cup \{[j, 2\ell - 1 + j, \infty, 3\ell - 1 + j; 4\ell - 1 + j]_{C4i} \mid j = 0, 1, \ldots, 5\ell - 2\}$.

For an X_{1i} decomposition of D_v for $v = 10\ell + 1$, consider: $\{[j, 2 + 5i + j, 4 + 10i + j, 3 + 5i + j; 10\ell - 10i + j]_{X1i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell\} \cup \{[j, 10\ell - 3 - 5i + j, 10\ell + j, 10\ell - 4 - 5i + j; 10\ell - 5 - 10i + j]_{X1i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell\}$.

For an X_{1i} decomposition of D_v for $v = 10\ell + 6$, consider: $\{[j, 2 + 5i + j, 4 + 10i + j, 3 + 5i + j; 10\ell + 5 - 10i + j]_{X1i} \mid i = 0, 1, \ldots, \ell, j = 0, 1, \ldots, 10\ell + 5\} \cup \{[j, 10\ell + 2 - 5i + j, 10\ell + 5 + j, 10\ell + 1 - 5i + j; 10\ell - 10i + j]_{X1i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell + 5\}$.

Case 3. For an X_{2i} decomposition of D_v for $v = 5\ell$, $\ell \geq 2$, consider: $\{[j, 1 + i + j, 4\ell + j, \ell + 1 + i + j; \ell - i + j]_{X2i} \mid i = 0, 1, \ldots, \ell - 2, j = 0, 1, \ldots, 5\ell - 2\} \cup \{[j, 2\ell - 1 + j, 4\ell - 1 + j, \infty; 1 + j]_{X2i} \mid j = 0, 1, \ldots, 5\ell - 2\}$.

For an X_{2i} decomposition of D_v for $v = 10\ell + 1$, consider: $\{[j, 10\ell - 1 - 5i + j, 10\ell - 3 - 10i + j, 10\ell - 2 - 5i + j; 1 + 10i + j]_{X2i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell\} \cup \{[j, 4 + 5i + j, 8 + 10i + j, 3 + 5i + j; 6 + 10i + j]_{X2i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell\}$.

For an X_{2i} decomposition of D_v for $v = 10\ell + 6$, consider: $\{[j, 10\ell + 4 - 5i + j, 10\ell + 2 - 10i + j, 10\ell + 3 - 5i + j; 1 + 10i + j]_{X2i} \mid i = 0, 1, \ldots, \ell, j = 0, 1, \ldots, 10\ell + 5\} \cup \{[j, 4 + 5i + j, 8 + 10i + j, 3 + 5i + j; 6 + 10i + j]_{X2i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell + 5\}$.

Case 4. For an X_{3i} decomposition of D_v for $v = 5\ell$, consider: If $\ell = 1$ then take $\{[j, \infty, 1 + j, 2 + j; 3 + j]_{X3i} \mid j = 0, 1, 2, 3\}$ and if $\ell \geq 2$ then take $\{[j, 2\ell + i + j, \ell - 1 + j, 5\ell - 2 - i + j; \ell - i + j]_{X3i} \mid i = 0, 1, \ldots, \ell - 3, j = 0, 1, \ldots, 5\ell - 2\} \cup \{[j, 3\ell - 2 + j, \ell - 1 + j, 4\ell + j; 3\ell + j]_{X3i}, [j, \infty, 5\ell - 2 + j, 1 + j; 2\ell + j]_{X3i} \mid j = 0, 1, \ldots 5\ell - 2\}$.

For an X_{3i} decomposition of D_v for $v = 10\ell + 1$, consider: $\{[j, 1 + 5i + j, 4 + 10i + j, 2 + 5i + j; 10\ell - 4 - 10i + j]_{X3i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell\} \cup$ $[j, 10\ell - 4 - 5i + j, 10\ell - 7 - 10i + j, 10\ell - 3 - 5i + j; 10\ell - 9 - 10i + j]_{X3i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell\}$.

For an X_{3i} decomposition of D_v for $v = 10\ell + 6$, consider: $\{[j, 1 + 5i + j, 4 + 10i + j, 2 + 5i + j; 10\ell + 1 - 10i + j]_{X3i} \mid i = 0, 1, \ldots, \ell, j = 0, 1, \ldots, 10\ell + 5\} \cup \{[j, 10\ell + 1 - 5i + j, 10\ell - 2 - 10i + j, 10\ell + 2 - 5i + j; 10\ell - 4 - 10i + j]_{X3i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots 10\ell + 5\}$.

Case 5. For an X_{4i} decomposition of D_v for $v = 5\ell$, consider: If $\ell = 1$ then take $\{[j, 3 + j, \infty, 1 + j; 2 + j]_{X4i} \mid j = 0, 1, 2, 3\}$, if $\ell = 2$ then take $\{[j, 8 + j, 4 + j, 7 + j; 5 + j]_{X4i}, [j, 2 + j, \infty, 6 + j; 8 + j]_{X4i} \mid j = 0, 1, \ldots, 8\}$, and if $\ell \geq 3$ then take $\{[j, \ell + 1 + i + j, \ell + j, 4\ell - 1 - i + j; 5\ell - 2 - i + j]_{X4i} \mid$

$i = 0, 1, \ldots, \ell - 3, j = 0, 1, \ldots, 5\ell - 2\} \cup \{[j, 4\ell + j, \ell + 2 + j, 3\ell + 1 + j; 4\ell - 1 + j]_{X4i}, [j, 2\ell + j, \infty, 3\ell + j; 4\ell + j]_{X4i} \mid j = 0, 1, \ldots, 5\ell - 2\}$.

For an X_{4i} decomposition of D_v for $v = 10\ell + 1$, consider: $\{[j, 2 + 5i + j, 1 + j, 10\ell - 1 - 5i + j; 10\ell - 10i + j]_{X4i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell\} \cup \{[j, 3 + 5i + j, 10\ell + j, 4 + 5i + j; 10\ell - 5 - 10i + j]_{X4i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 10\ell\}$.

For an X_{4i} decomposition of D_v for $v = 10\ell + 6$, consider: $\{[j, 2 + 5i + j, 1 + j, 10\ell + 4 - 5i + j; 10\ell + 5 - 10i + j]_{X4i} \mid i = 0, 1, \ldots, \ell, j = 0, 1, \ldots, 10\ell + 5\} \cup \{[j, 3 + 5i + j, 10\ell + 5 + j, 4 + 5i + j; 10\ell - 10i + j]_{X4i} \mid i = 0, 1, \ldots \ell - 1, j = 0, 1, \ldots, 10\ell + 5\}$.

Case 6. For anY_{1i} decomposition of D_v for $v = 5\ell$, $v \neq 5$, consider: $\{[j, 5\ell - 2 - 5i + j, 5\ell - 7 - 10i + j, 5\ell - 5 - 5i + j; 5\ell - 4 - 5i + j]_{Y1i} \mid i = 0, 1, \ldots, \ell - 2, j = 0, 1, \ldots, 5\ell - 2\} \cup \{[j, \infty, 4 + j, 3 + j; 2 + j]_{Y1i} \mid j = 0, 1, \ldots, 5\ell - 2\}$.

For a Y_{1i} decomposition of D_v for $v = 5\ell + 1$, $v \neq 6$, consider: $\{[j, 5\ell - 1 - 5i + j, 5\ell - 6 - 10i + j, 5\ell - 3 - 5i + j; 5\ell - 5i + j]_{Y1i} \mid i = 0, 1, \ldots, \ell - 3, j = 0, 1, \ldots, 5\ell\} \cup \{[j, 4 + j, 5 + j, 2 + j; 10 + j]_{Y1i} \mid j = 0, 1, \ldots, 5\ell\} \cup \{[j, 9 + j, 15 + j, 7 + j; 5 + j]_{Y1i} \mid j = 0, 1, \ldots, 5\ell\}$.

Case 7. For a Y_{2i} decomposition of D_v for $v = 5\ell$, consider: $\{[j, 1 + 5i + j, 5\ell - 2 + j, 5\ell - 6 - 5i + j; 5\ell - 4 - 5i + j]_{Y2i} \mid i = 0, 1, \ldots, \ell - 2, j = 0, 1, \ldots 5\ell - 2\} \cup \{[j, 5\ell - 3 + j, \infty, 3 + j; 1 + j]_{Y2i} \mid j = 0, 1, \ldots, 5\ell - 2\}$.

For a Y_{2i} decomposition of D_v for $v = 5\ell + 1$, consider: $\{[j, 2 + 5i + j, 5\ell - 1 + j, 1 + 5i + j; 5 + 5i + j]_{Y2i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 5\ell\}$.

Case 8. For a Y_{3i} decomposition of D_v for $v = 5\ell$, consider: $\{[j, 2 + 5i + j, 7 + 10i + j, 4 + 5i + j; 3 + 5i + j]_{Y3i} \mid i = 0, 1, \ldots, \ell - 2, j = 0, 1, \ldots, 5\ell - 2\} \cup \{[j, 5\ell - 3 + j, 5\ell - 4 + j, \infty; 5\ell - 2 + j]_{Y3i} \mid j = 0, 1, \ldots, 5\ell - 2\}$.

For a Y_{3i} decomposition of D_v for $v = 5\ell + 1$, consider: $\{[j, \ell + 1 + 2i + j, \ell + j, \ell + 2 + 2i + j; 5\ell - i + j]_{Y3i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 5\ell\}$.

Case 9. For a Z_{1i} decomposition of D_v for $v \equiv 0 \pmod 5$, we present a recursive argument. First, to show that a Z_{1i} decomposition of D_{10} exists, suppose the vertex set of D_{10} is $\{0_1, 1_1, 2_1, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2\}$. Consider the blocks: $\{[0_2, 0_1, 4_2, 2_1; 1_2]_{Z1i}$, $[4_2, 2_2, 3_2, 1_2; 3_1]_{Z1i}$, $[0_2, 3_1, 1_2, 1_1; 4_1]_{Z1i}$, $[2_2, 2_1, 1_2, 4_1; 3_2]_{Z1i}$, $[1_2, 0_2, 4_2, 3_2; 0_1]_{Z1i}$, $[2_2, 0_1, 3_2, 3_1; 1_1]_{Z1i}$, $[3_2, 1_1, 4_2, 4_1; 2_1]_{Z1i}$, $[0_2, 2_2, 1_2, 4_2; 3_2]_{Z1i}$, $[2_2, 0_2, 3_2, 4_2; 1_2]_{Z1i}$, $[0_1, 0_2, 4_1, 2_2; 1_1]_{Z1i}$, $[4_1, 2_1, 3_1, 1_1; 3_2]_{Z1i}$, $[0_1, 3_2, 1_1, 1_2; 4_2]_{Z1i}$, $[2_1, 2_2, 1_1, 4_2; 3_1]_{Z1i}$, $[1_1, 0_1, 4_1, 3_1; 0_2]_{Z1i}$, $[2_1, 0_2, 3_1, 3_2; 1_2]_{Z1i}$, $[3_1, 1_2, 4_1, 4_2; 2_2]_{Z1i}$, $[0_1, 2_1, 1_1, 4_1; 3_1]_{Z1i}$, $[2_1, 0_1, 3_1, 4_1; 1_1]_{Z1i}\}$.

We now show that Z_{1i} decomposition of $D_{5\ell}$ exists for all $\ell \geq 2$. We do so recursively and establish this result by induction. We have seen that there is a decomposition for $\ell = 2$. Now suppose that such a decomposition exists for $\ell = k$. Then there exists a Z_{1i} decomposition of D_{5k} where the vertex set of D_{5k} is $V = \{0_1, 1_1, 2_2, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2, \ldots, 0_k, 1_k, 2_k, 3_k, 4_k\}$. Consider D_{5k+5} with vertex set $V \cup \{0_{k+1}, 1_{k+1}, 2_{k+1}, 3_{k+1}, 4_{k+1}\}$. Add to the decomposition of D_{5k} the following: $\{[0_{k+1}, 0_1, 4_{k+1}, 2_1; 1_{k+1}]_{Z1i}, [4_{k+1}, 2_{k+1}, 3_{k+1}, 1_{k+1}; 3_1]_{Z1i}, [0_{k+1}, 3_1, 1_{k+1}, 1_1; 4_1]_{Z1i}, [2_{k+1}, 2_1, 1_{k+1}, 4_1; 3_{k+1}]_{Z1i}, [1_{k+1}, 0_{k+1}, 4_{k+1}, 3_{k+1}; 0_1]_{Z1i}, [2_{k+1}, 0_1, 3_{k+1}, 3_1; 1_1]_{Z1i}, [3_{k+1}, 1_1, 4_{k+1}, 4_1; 2_1]_{Z1i}, [0_{k+1}, 2_{k+1}, 1_{k+1}, 4_{k+1}; 3_{k+1}]_{Z1i}, [2_{k+1}, 0_{k+1}, 3_{k+1}, 4_{k+1}; 1_{k+1}]_{Z1i}\} \cup \{[4_i, 2_j, 0_i, 0_j; 3_j]_{Z1i}, [0_i, 3_j, 1_i, 1_j; 4_j]_{Z1i}, [1_i, 4_j, 2_i, 2_j; 0_j]_{Z1i}, [2_i, 0_j, 3_i, 3_j; 1_j]_{Z1i}, [3_i, 1_j, 4_i, 4_j; 2_j]_{Z1i} \mid$ <u>either</u> $i = k+1$ and $j = 2, 3, \ldots, k$ <u>or</u> $j = k+1$ and $i = 2, 3, \ldots, k\}$.

For a Z_{1i} decomposition of D_v for $v = 5\ell + 1$, consider: $\{[j, 2+5i+j, 5\ell-1+j, 3+5i+j; 1+5i+j]_{Z1i} \mid i = 0, 1, \ldots, \ell-1, j = 0, 1, \ldots, 5\ell\}$. Here, reduce vertex labels modulo v.

Case 10. For a Z_{2i} decomposition of D_v for $v \equiv 0 \pmod 5$, we present a recursive argument. First, to show that a Z_{2i} decomposition of D_{10} exists, suppose the vertex set of D_{10} is $\{0_1, 1_1, 2_1, 2_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2\}$. Consider the blocks: $\{[1_2, 0_2, 3_2, 4_2; 2_2]_{Z2i}, [0_2, 1_2, 3_2, 2_2; 4_2]_{Z2i}, [1_2, 2_2, 4_2, 3_2; 3_1]_{Z2i}, [0_2, 3_2, 2_2, 4_2; 2_1]_{Z2i}, [0_2, 1_1, 4_2, 4_1; 2_2]_{Z2i}, [1_2, 2_1, 0_2, 0_1; 4_2]_{Z2i}, [2_2, 3_1, 1_2, 1_1; 4_1]_{2i}, [3_2, 4_1, 2_2, 2_1; 0_1]_{Z2i}, [4_2, 0_1, 3_2, 3_1; 1_1]_{2i}, \{[1_1, 0_1, 3_1, 4_1; 2_1]_{Z2i}, [0_1, 1_1, 3_1, 2_1; 4_1]_{Z2i}, [1_1, 2_1, 4_1, 3_1; 3_2]_{Z2i}, [0_1, 3_1, 2_1, 4_1; 2_2]_{Z2i}, [0_1, 1_2, 4_1, 4_2; 2_1]_{Z2i}, [1_1, 2_2, 0_1, 0_2; 4_1]_{Z2i}, [2_1, 3_2, 1_1, 1_2; 4_2]_{2i}, [3_1, 4_2, 2_1, 2_2; 0_2]_{Z2i}, [4_1, 0_2, 3_1, 3_2; 1_2]_{2i}\}$

We now show that Z_{2i} decomposition of $D_{5\ell}$ exists for all $\ell \geq 2$. We do so recursively and establish this result by induction. We have seen that there is a decomposition for $\ell = 2$. Now suppose that such a decomposition exists for $\ell = k$. Then there exists a Z_{2i} decomposition of D_{5k} where the vertex set of D_{5k} is $V = \{0_1, 1_1, 2_2, 3_1, 4_1, 0_2, 1_2, 2_2, 3_2, 4_2, \ldots, 0_k, 1_k, 2_k, 3_k, 4_k\}$. Consider D_{5k+5} with vertex set $V \cup \{0_{k+1}, 1_{k+1}, 2_{k+1}, 3_{k+1}, 4_{k+1}\}$. Add to the decomposition of D_{5k} the following: $\{[1_{k+1}, 0_{k+1}, 3_{k+1}, 4_{k+1}; 2_{k+1}]_{Z2i}, [0_{k+1}, 1_{k+1}, 3_{k+1}, 2_{k+1}; 4_{k+1}]_{Z2i}, [1_{k+1}, 2_{k+1}, 4_{k+1}, 3_{k+1}; 3_1]_{Z2i}, [0_{k+1}, 3_{k+1}, 2_{k+1}, 4_{k+1}; 2_1]_{Z2i}, [0_{k+1}, 1_1, 4_{k+1}, 4_1; 2_{k+1}]_{Z2i}, [1_{k+1}, 2_1, 0_{k+1}, 0_1; 4_{k+1}]_{Z2i}, [2_{k+1}, 3_1, 1_{k+1}, 1_1; 4_1]_{2i}, [3_{k+1}, 4_1, 2_{k+1}, 2_1; 0_1]_{Z2i}, [4_{k+1}, 0_1, 3_{k+1}, 3_1; 1_1]_{2i} \cup \{[0_i, 0_j, 3_i, 4_j; 2_j]_{Z1i}, [1_i, 1_j, 4_i, 0_j; 3_j]_{Z1i}, [2_i, 2_j, 0_i, 1_j; 4_j]_{Z1i}, [3_i, 3_j, 1_i, 2_j; 0_j]_{Z1i}, [4_i, 4_j, 2_i, 3_j; 1_j]_{Z1i} \mid$ <u>either</u> $i = k+1$ and $j = 2, 3, \ldots, k$ <u>or</u> $j = k+1$ and $i = 2, 3, \ldots, k\}$.

For a Z_{2i} decomposition of D_v for $v = 5\ell + 1$, consider: $\{[j, 5\ell - 1 - 5i + j, 2 + j, 5\ell - 2 - 5i + j; 5\ell - 4 - 5i + j]_{Z2i} \mid i = 0, 1, \ldots, \ell - 1, j = 0, 1, \ldots, 5\ell\}$. Here, reduce vertex labels modulo v.

In each case, the presented blocks form a decomposition of D_v. Corresponding decompositions of D_v into each of the converses of the digraphs of Cases 1–10 immediately follow. ■

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Appendix

In this appendix, we give an argument that no X_{2i} decomposition of D_5 exists. Notice that the only vertex of $[a, b, c, d; e]_{X2i}$ which is of odd in-degree is d. So if $B_1 = [a_1, b_1, c_1, x; e_1]_{X2i}$ is a block of such a decomposition then, since vertex x has in-degree 4 (even) in D_5, another block of the decomposition must be of the form $B_2 = [a_2, b_2, c_2, x; e_2]_{X2i} \neq B_1$. Now let the vertices of D_5 be v_1, v_2, v_3, v_4, v_5 and let the blocks of a decomposition be B_1, B_2, B_3, B_4. Without loss of generality, $B_1 = [v_1, v_2, v_3, v_4; v_5]_{X2i}$. By the above argument, we know that another block of the decomposition (say block B_2) is of the form $B_2 = [a_2, b_2, c_2, v_4; e_2]_{X2i}$ where $\{a_2, b_2, c_2, e_2\} = \{v_1, v_2, v_3, v_5\}$. Notice that there are 4! ways to assign vertices v_1, v_2, v_3, v_5 to the positions a_2, b_2, c_2, e_2 in block B_2. Now if block B_3 is of the form $[a_3, b_3, c_3, v_i; e_3]_{X2i}$ then block B_4 must be of the form $[a_4, b_4, c_4, v_i; e_4]_{X2i}$, again by the above argument. There are 5 ways to assign a vertex to vertex v_i in blocks B_3 and B_4. There are then 4! ways to assign the remaining vertices to positions a_3, b_3, c_3, e_3 in block B_3 and 4! ways to assign the remaining vertices to positions a_4, b_4, c_4, e_4 in block B_4. Hence, there are $4! \times 4! \times 4! \times 5 = 69,120$ different ways to assign the 5 vertices of to blocks B_1, B_2, B_3, B_4. If such a decomposition exists, then it must be one of these possibilities. The authors have written a program to test these possibilities and none of them yield a decomposition. (The authors readily admit that there must be a more elegant argument for this—D_5 and X_{2i} each contain 5 vertices, so certainly reducing this to 69,120 cases and then beating them death cannot be the best method!)