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Tricyclic Steiner Triple Systems

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Abstract A Steiner triple system of order v, denoted $STS(v)$, is said to be *tricyclic* if it admits an automorphism whose disjoint cyclic decomposition consists of three cycles. In this paper we give necessary and sufficient conditions for the existence of a tricyclic $STS(v)$ for several cases. We also pose conjectures concerning their existence in two remaining cases.

Keywords Steiner triple systems · Tricyclic automorphism

1 Introduction

A *Steiner triple system* of order v, denoted $STS(v)$, is a v-element set, X, of points, together with a set β, of unordered triples of elements of X, called *blocks*, such that any two points of X are together in exactly one block of β. It is well known that a $STS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6). For a general review of triple systems in general, see [5]. An *automorphism* of a STS is a permutation π of X which fixes β. A permutation π of a v-element set is said to be of *type* $[\pi] = [\pi_1, \pi_2, \ldots, \pi_v]$

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if the disjoint cyclic decomposition of π contains π_i cycles of length i (therefore $\sum_{i=1}^{v} i\pi_i = v$). The *orbit* of a block under an automorphism π is the image of the block under the powers of π. A collection of blocks B is said to be a *collection of base blocks for a STS under the permutation* π if the orbits of the blocks of B produce the STS and exactly one block of B occurs in each orbit.

Several types of automorphisms have been explored in connection with the question "For which orders v does there exist a $STS(v)$ admitting an automorphism of the given type?" A *cyclic* $STS(v)$ is one admitting an automorphism of type $[0, 0, \ldots, 0, 1]$ and exists if and only if $v \equiv 1$ or 3 (mod 6) and $v \neq 9$ [8]. A k-rotational $STS(v)$ admits an automorphism of type $[1, 0, 0, \ldots, 0, k, 0, \ldots, 0]$. A 1-rotational $STS(v)$ exists if and only if $v \equiv 3$ or 9 (mod 24) and a 2-rotational $STS(v)$ exists if and only if $v \equiv 1, 3, 7, 9, 15$ or 19 (mod 24) [9]. A 1-rotational Steiner triple system can also be viewed as a STS which admits an automorphism group that acts sharply transitively on all but one point [1]. A k-transrotational $STS(v)$ admits an automorphism of type $[1, 1, 0, 0, \ldots, 0, k, 0, 0, 0]$ and with $k = 1$ such a system exists if and only if $v \equiv 1, 7, 9$ or 15 (mod 24) [7]. A *bicyclic* $STS(v)$ admits an automorphism of type $[\pi] = [0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_M = \pi_N = 1$, $M < N$ and $M + N = v$. That is, the disjoint cyclic decomposition of π consists of one cycle of length M and another (larger) cycle of length N. Such a bicyclic $STS(v)$ with $M > 1$ exists if and only if $M \equiv 1$ or 3 (mod 6), $M \neq 9$, $M \mid N$, and $v = M+N \equiv 1$ or 3 (mod 6) [2,3,6]. Notice that a 1-rotational STS is a special case of a bicyclic STS, but the existence of 1-rotational STSs does not fit the same pattern as bicyclic STSs (i.e., we cannot simply take $M = 1$ in the previously stated conditions for bicyclic STSs to get the conditions for 1-rotational STSs). Another related structure is a reverse $STS(v)$. Such a structure admits an automorphism of type $[\pi] = [1, (v-1)/2, 0, \ldots, 0]$ and exists if and only if $v \equiv 1, 3, 9$, or 19 (mod 24) [10].

The purpose of this paper is to explore several categories of Steiner triple systems which admit an automorphism consisting of three disjoint cycles. Therefore we define a *tricyclic* $STS(v)$ to be one that admits an automorphism either of type $[0, \ldots, 0, 3, 0, \ldots, 0]$, $[0, \ldots, 0, 1, 0, \ldots 0, 2, 0, \ldots, 0]$, or of type $[0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$. We classify tricyclic Steiner triple systems admitting an automorphism of the first two types in Sect. 2. In Sect. 3, we give necessary and sufficient conditions for the existence of a Steiner triple system admitting an automorphism of the third type in the case when the smallest cycle of the automorphism is of length one. In Sect. 4, we give two conjectures concerning the remaining cases for the third type of automorphism.

2 Some Special Tricyclic Steiner Triple Systems

From the existence of a cyclic $STS(v)$ we readily have:

Theorem 1 *A tricyclic $STS(v)$ admitting an automorphism of type* $[0, \ldots, 0, 3, 0, \ldots, 0]$ *exists if and only if* $v \equiv 3$ *(mod 6).*

Proof Of course the condition $v \equiv 3$ (mod 6) is necessary. For all such v, except $v = 9$, there is a cyclic $STS(v)$. Simply by cubing the cyclic automorphism, we see that the systems are also tricyclic. The affine plane over $\mathbb{Z}_3$ is equivalent to a $STS(9)$ and every translation of this affine plane is obviously an automorphism of the required type for the $STS(9)$. □

Similarly, we can establish the existence of a large class of tricyclic STSs from the existence of the bicyclic STSs. Here, and throughout, we represent the ordered pair (x, y) as the subscripted pair x_y.

Theorem 2 *A tricyclic* $STS(v)$ *admitting an automorphism of type* $[\pi] = [0, \ldots, 0, 1, 0, \ldots, 0, 2, 0, \ldots, 0]$ *where* $\pi_M = 1$, $\pi_N = 2$ *and* $M > 1$ *exists if and only if* $M \equiv 1$ *or* 3 (*mod* 6), $M \neq 9$, $M \mid N$ *and* $v = M + 2N \equiv 1$ *or* 3 (*mod* 6).

Proof First, suppose there is such a system with the point set $\mathbb{Z}_M \bigcup \mathbb{Z}_N \times \mathbb{Z}_2$ admitting the automorphism $\pi = (0, 1, \ldots, M-1)(0_0, 1_0, \ldots, (N-1)_0)(0_1, 1_1, \ldots, (N-1)_1)$. It is rather easy to see that the fixed points of an automorphism form a subsystem of a STS (i.e., if two points of a triple are fixed by the automorphism, then the third point of the triple must also be fixed by the automorphism). By considering π^M we see, therefore, that such a $STS(v)$ has a cyclic subsystem of order M. Therefore, $M \equiv 1$ or 3 (mod 6) and $M \neq 9$ is necessary. Also, such a STS must contain some block of the form (x, y_i, z_j) where $x \in \mathbb{Z}_M$ and $y_i, z_j \in \mathbb{Z}_N \times \mathbb{Z}_2$. By applying π^N to this block, we see that $(\pi^N(x), y_i, z_j)$ must also be a block of the STS and therefore $\pi^N(x) = x$ and $M \mid N$ is necessary.

To establish sufficiency, suppose M and N satisfy the stated conditions. Then there is a bicyclic $STS(v)$ admitting an automorphism consisting of a cycle of length M and a cycle of length $2N$. By considering the square of this automorphism, we see that the bicyclic $STS(v)$ is also tricyclic and admits an automorphism of the desired type. □

Notice that 2-rotational and 1-transrotational STSs are also examples of tricyclic STSs.

3 Tricyclic Steiner Triple Systems for Which the Smallest Cycle is of Length One

We now turn our attention to STSs admitting automorphisms of type $[\pi] = [1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_1 = \pi_M = \pi_N = 1$, $v = M + N + 1$, and $M < N$. In our discussion, we will let the point set of such a system be $\{\infty\} \bigcup \mathbb{Z}_M \times \{0\} \bigcup \mathbb{Z}_N \times \{1\}$ and let the automorphism be $\pi = (\infty)(0_0, 1_0, \ldots, (M-1)_0)(0_1, 1_1, \ldots, (N-1)_1)$. As in the proof of Theorem 2, by considering π^M, we see that the $STS(v)$ contains a 1-rotational subsystem of order $M + 1$. Therefore we have:

Lemma 1 *If a tricyclic* $STS(v)$ *exists admitting an automorphism of the type* $[\pi] = [1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ *where* $\pi_1 = \pi_M = \pi_N = 1$ *then* $M \equiv 2$ *or* 8 (*mod* 24).

Also, in such a STS there must be some block of the form (x_0, y_1, z_1). By considering the image of this block under π^N, as in Theorem 2, we have:

Lemma 2 *If a tricyclic* $STS(v)$ *as described in* Lemma 1 *exists, then* $M \mid N$.

With a pair of points of the form (x_1, y_1) we associate a *pure difference of type* 1 of $\min\{(x-y)(\text{mod } N), (y-x)(\text{mod } N)\}$. With a pair of points of the form (x_0, y_1) we associate the *mixed difference* $(y-x)$ (mod M). The set of pure differences of type 1 is $\{1, 2, \ldots, N/2\}$ and the set of mixed differences is $\{0, 1, \ldots, M-1\}$. A collection of base blocks for the desired type of STS must contain a block of the form $(\infty, x_1, (x+N/2)_1)$. Notice that this block contains a pair of points with the associated pure difference of type 1 of $N/2$. Therefore, constructing the desired type of STS is equivalent to presenting a collection of blocks on the point set $\mathbb{Z}_M \times \{0\} \bigcup \mathbb{Z}_N \times \{1\}$ such that the differences associated with the pairs of points of these blocks precisely cover the set of pure differences of type 1 of $\{1, 2, \ldots, N/2-1\}$ and the set of mixed differences of $\{0, 1, \ldots, M-1\}$. Such a collection of blocks along with a collection of base blocks for a 1-rotational $STS(M+1)$ on the point set $\{\infty\} \bigcup \mathbb{Z}_M \times \{1\}$ (under the obvious automorphism) and the block $(\infty, 0_1, (N/2)_1)$ form a collection of base blocks for a tricyclic $STS(1+M+N)$ with a 1-rotational subsystem under π.

We have a final necessary condition:

Lemma 3 *If a tricyclic* $STS(v)$ *as described in* Lemma 3.1 *exists, then* $N = kM$ *where* $k \equiv 2, 3, 6$ *or* 11 (*mod* 12) *whenever* $M \equiv 2$ (*mod* 24). *If* $M \equiv 8$ (*mod* 24), *then* $k \equiv 0$ *or* 2 (*mod* 3).

Proof A base block of the form (x_0, y_1, z_1) covers two mixed differences and one pure difference of type 1. One of the mixed differences must be congruent to the sum of the other two differences modulo M. Since M is even, either zero or two of these differences is/are odd. If $3 \mid N$, then a possible base block is one of the form $(x_1, (x+N/3)_1, (x+2N/3)_1)$. A block of this type is said to be a *short orbit block* since the length of its orbit under π is precisely one-third the length of the orbit of any other block on the points $\mathbb{Z}_N \times \{1\}$. A short orbit block covers the pure difference of type 1 of $N/3$ only, and $N/3$ is even. A base block of the form (x_1, y_1, z_1) (other than a short orbit block) covers three distinct pure differences of type 1. These three differences satisfy either the condition that one is the sum of the other two, or the condition that all three sum to 0 modulo N. In either case, either zero or two of these differences is/are odd. So, a collection of blocks of the form (x_0, y_1, z_1) or (x_1, y_1, z_1) covers an even number of odd differences. Therefore, the number of odd differences in the set $\{0, 1, \ldots, M-1\} \bigcup \{1, 2, \ldots, N/2-1\}$ must be even. From this, the lemma follows. □

We now show that the necessary conditions of Lemmas 1–3 are sufficient in a series of constructions.

Lemma 4 *If* $M \equiv 2$ (*mod* 24) *and* $k \equiv 2, 3, 6$ *or* 11 (*mod* 12), *then there exists a tricyclic* $STS(v)$ *as described above.*

Proof Consider the given collections of blocks.
Case 1. Suppose that $M \equiv 2$ (mod 24) and $k \equiv 2$ (mod 12).
If $M = 26$ and $k = 2$, consider the following collection of blocks:

$(0_1, 7_1, 18_1), (0_1, 8_1, 17_1), (0_1, 13_1, 25_1), (0_1, 14_1, 24_1), (\infty, 0_1, 26_1),$

$(0_0, 0_1, 15_1)$, $(0_0, 1_1, 17_1)$, $(0_0, 8_1, 28_1)$, $(0_0, 7_1, 29_1)$, $(0_0, 11_1, 30_1)$, $(0_0, 10_1, 31_1)$,
$(0_0, 9_1, 32_1)$, $(0_0, 12_1, 18_1)$, $(0_0, 13_1, 16_1)$, $(0_0, 14_1, 19_1)$, $(0_0, 20_1, 24_1)$, $(0_0, 23_1, 25_1)$, $(0_0, 21_1, 22_1)$.

Otherwise, consider the following collection of blocks:

$(0_1, (((k-1)M+10)/6-2r)_1, ((k-1)M/2)-r)_1)$ for $r = 1, 2, \dots, ((k-1)M-2)/12$,
$(0_1, (((k-1)M-8)/6-2r)_1, (((k-1)M-5)/3)-r)_1)$ for $r = 1, 2, \dots, ((k-1)M-50)/24$ (omit if $M = 26$),
$(0_1, (((k-1)M-14)/12-2r)_1, ((7(k-1)M-14)/24-r)_1)$ for $r = 1, 2, \dots, ((k-1)M-50)/24$,
$(0_1, (((k-1)M+10)/12)_1, ((7(k-1)M-14)/24)_1)$, $(0_1, 1_1, ((5(k-1)M+14)/24)_1)$,
$(0_1, (((k-1)M-14)/12)_1, (((k-1)M-5)/3)_1)$ (omit if $M = 26$),
$(0_1, (((k-1)M-8)/6)_1, ((5(k-1)M-10)/12)_1)$, $(\infty, 0_1, (kM/2)_1)$,
$(0_0, (M-r)_1, (((k+1)M-2)/2+r)_1)$ for $r = 1, 2, \dots, (M-2)/4$,
$(0_0, ((M-2)/2)-r)_1, ((kM-2)/2+r)_1)$ for $r = 1, 2, \dots, (M-2)/8$,
$(0_0, ((3M-6)/8-r)_1, (((4k+1)M+6)/8+r)_1)$ for $r = 1, 2, \dots, (M-18)/8$,
$(0_0, ((M+6)/8)_1, (((4k+1)M-2)/8)_1)$, $(0_0, ((3M-2)/4)_1, (((2k+1)M-2)/4)_1)$, and
$(0_0, ((M-2)/2)_1, (((2k+1)M+2)/4)_1)$.

Case 2. Suppose that $M \equiv 2 \pmod{24}$ and $k \equiv 3 \pmod{12}$. Consider the following collection of blocks:

$(0_1, (kM/3)_1, (2kM/3)_1)$, $(\infty, 0_1, (kM/2)_1)$,
$(0_1, (kM/6-2r)_1, (kM/2-r)_1)$ for $r = 1, 2, \dots, ((k-1)M-4)/12$,
$(0_1, ((kM+6)/6-2r)_1, (kM/3-r)_1)$ for $r = 1, 2, \dots, ((k-1)M-4)/12$,
$(0_0, ((M+4)/3-r)_1, (((3k+3)M-12)/12+r)_1)$ for $r = 1, 2, \dots, (M+10)/12$,
$(0_0, ((M+1)/3+r)_1, (((3k+5)M+8)/12-r)_1)$ for $r = 1, 2, \dots, (M-2)/12$,
$(0_0, (((3k+5)M-4)/12+r)_1, (((8k+6)M+12)/12-r)_1)$ for $r = 1, 2, \dots, (M-2)/12$,
$(0_0, ((4k+3)M/6+r)_1, (((13k+7)M+4)/12-r)_1)$ for $r = 1, 2, \dots, (M-2)/24$,
$(0_0, (((16k+13)M-2)/24+r)_1, (((16k+17)M+14)/24-r)_1)$ for $r = 1, 2, \dots, (M-2)/12$,
$(0_0, (((16k+17)M-10)/24+r)_1, (((16k+21)M+30)/24-r)_1)$ for $r = 1, 2, \dots, (M-2)/24$,
$(0_0, (((16k+18)M-12)/24+r)_1, ((26k+18)M/24-r)_1)$ for $r = 1, 2, \dots, (M-2)/24$,
$(0_0, (((26k+15)M+6)/24+r)_1, (((26k+17)M+2)/24-r)_1)$ for $r = 1, 2, \dots, (M-26)/24$,
$(0_0, (((13k+7)M+4)/12)_1, (((26k+16)M+4)/24)_1)$.

Case 3. Suppose that $M \equiv 2 \pmod{24}$ and $k \equiv 6 \pmod{12}$. Consider the following collection of blocks:

$(0_1, (kM/4)_1, (5kM/12)_1)$, $(0_1, (kM/3)_1, (2kM/3)_1)$,
$(0_1, (((k+1)M-8)/6+r)_1, (((2k-1)M+8)/6-r)_1)$ for $r = 1, 2, \ldots, ((k-2)M+4)/12$,
$(0_1, (kM/3+r)_1, ((kM-2)/2-r)_1)$ for $r = 1, 2, \ldots, (kM-24)/12$,
$(\infty, 0_1, (kM/2)_1)$, $(0_0, 0_1, (5kM/12-1)_1)$,
$(0_0, ((M-2)/2+r)_1, (((k+3)M-6)/6-r)_1)$ for $r = 1, 2, \ldots, (M-8)/6$,
$(0_0, ((M-5)/3)_1, ((M-2)/3)_1)$, $(0_0, ((M-6)/4)_1, (((4k+1)M-2)/12)_1)$,
$(0_0, ((M-6)/4-r)_1, (((2k+3)M-18)/12+r)_1)$ for $r = 1, 2, \ldots, (M-14)/12$,
$(0_0, ((M-2)/12+r)_1, (((4k+1)M-2)/12-r)_1)$ for $r = 1, 2, \ldots, (M-14)/12$,
$(0_0, ((9M-18)/12-r)_1, (((2k+9)M-30)/12+r)_1)$ for $r = 1, 2, \ldots, (M-2)/12$,
$(0_0, ((11M-46)/12+r)_1, (((4k+11)M-34)/12-r)_1)$ for $r = 1, 2, \ldots, (M-14)/12$,
$(0_0, (M-3)_1, (((k+6)M-24)/6)_1)$, and $(0_0, (M-1)_1, (((k+2)M-4)/2)_1)$.

Case 4. Suppose that $M \equiv 2 \pmod{24}$ and $k \equiv 11 \pmod{12}$. Consider the following collection of blocks:

$(0_1, (((k-1)M-8)/6-2r)_1, (((k-1)M-2)/2-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-20)/12$,
$(0_1, (((k-1)M-2)/6-2r)_1, (((k-1)M-2)/3-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-44)/24$,
$(0_1, (((k-1)M-8)/12-2r)_1, ((7(k-1)M+4)/24-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-44)/24$,
$(0_1, (((k-1)M+16)/12)_1, ((7(k-1)M+4)/24)_1)$, $(0_1, 1_1, ((5(k-1)M+20)/24)_1)$,
$(0_1, (((k-1)M-8)/12)_1, (((k-1)M-2)/3)_1)$, $(0_1, (((k-1)M-8)/6)_1, ((5(k-1)M-4)/12)_1)$,
$(0_1, (((k-1)M-2)/6)_1, ((k-1)M/2)_1)$, $(\infty, 0_1, (kM/2)_1)$,
$(0_0, 0_1, (((k-1)M-2)/2)_1)$, $(0_0, ((M-10)/8)_1, (((4k-1)M-6)/8)_1)$,
$(0_0, ((M-2)/8)_1, (((4k-3)M+6)/8)_1)$, $(0_0, ((M-2)/4)_1, ((kM-4)/2)_1)$,
$(0_0, ((M+2)/4)_1, (((2k+1)M-6)/4)_1)$,
$(0_0, ((M-4)/2+r)_1, (((k+1)M-2)/2-r)_1)$ for $r = 1, 2, \ldots, (M-2)/4$,
$(0_0, r_1, ((kM-4)/2-r)_1)$ for $r = 1, 2, \ldots, (M-18)/8$, and
$(0_0, ((M+6)/8+r)_1, (((4k-1)M-6)/8-r)_1)$ for $r = 1, 2, \ldots, (M-18)/8$.

In each case, the given collection of blocks, along with a collection of base blocks for a 1-rotational $STS(M+1)$ on the point set $\{\infty\} \bigcup \mathbb{Z}_M \times \{0\}$ under the automorphism $(\infty)(0_0, 1_0, \ldots, (M-1)_0)$, forms a collection of base blocks for a STS of the desired type. □

Lemma 5 *If $M \equiv 8 \pmod{24}$ and $k \equiv 0$ or $2 \pmod 3$, then there exists a tricyclic $STS(v)$ as described above.*

Proof Consider the given collections of blocks.
Case 1. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 0 \pmod{12}$. Consider the following collection of blocks:

$(0_1, (kM/6+r)_1, (kM/3-r)_1)$ for $r = 1, 2, \ldots, (kM-12)/12$,

$(0_1, (((2k+1)M-8)/6+r)_1, (((3k-1)M+2)/6-r)_1)$ for $r = 1, 2, \ldots, ((k-2)M-8)/12$,
$(0_1, (kM/4)_1, (5kM/12)_1)$, $(0_1, (kM/3)_1, (2kM/3)_1)$, $(\infty, 0_1, (kM/2)_1)$,
$(0_0, ((M-8)/6)_1, ((M-2)/6)_1)$, $(0_0, ((7M-20)/12)_1, (((6k+5)M-16)/12)_1)$,
$(0_0, ((M-2)/6+r)_1, (((k+1)M-8)/6-r)_1)$ for $r = 1, 2, \ldots, (M-8)/6$,
$(0_0, ((5M-16)/12-r)_1, (((4k+5)M-16)/12+r)_1)$ for $r = 1, 2, \ldots, (M-8)/12$,
$(0_0, ((7M-20)/12+r)_1, (((6k+7)M-20)/12-r)_1)$ for $r = 1, 2, \ldots, (M-8)/12$,
$(0_0, ((8M-28)/12+r)_1, (((4k+10)M-32)/12-r)_1)$ for $r = 1, 2, \ldots, (M-8)/12$,
$(0_0, ((11M-40)/12+r)_1, (((6k+11)M-52)/12-r)_1)$ for $r = 1, 2, \ldots, (M-20)/12$,
$(0_0, ((11M-40)/12)_1, (((2k+11)M-52)/12)_1)$, $(0_0, (M-3)_1, (((k+2)M-8)/2)_1)$, and
$(0_0, (M-1)_1, (((5k+12)M-24)/12)_1)$.

Case 2. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 2 \pmod 6$. Consider the following collection of blocks:

$(0_1, (((k-1)M+4)/6-2r)_1, ((k-1)M/2-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-8)/12$,
$(0_1, (((k-1)M-2)/6-2r)_1, (((k-1)M-2)/3-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-8)/24$,
$(0_1, (((k-1)M-20)/12-2r)_1, ((7(k-1)M-8)/24-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-56)/24$ (omit if $M = 8$),
$(0_1, 1_1, ((5(k-1)M+32)/24)_1)$, $(0_1, (((k-1)M-2)/6)_1, ((5(k-1)M-4)/12)_1)$,
$(0_1, (((k-1)M-20)/12)_1, (((k-1)M-2)/3)_1)$, $(\infty, 0_1, (kM/2)_1)$,
$(0_0, (M-r)_1, (((k+1)M-4)/2+r)_1)$ for $r = 1, 2, \ldots, M/4$,
$(0_0, ((M-4)/2-r)_1, ((kM-2)/2+r)_1)$ for $r = 1, 2, \ldots, (M-16)/8$,
$(0_0, ((3M-8)/8-r)_1, ((4k+1)M/8+r)_1)$ for $r = 1, 2, \ldots, (M-16)/8$,
$(0_0, (M/8)_1, (((4k+1)M-8)/8)_1)$, $(0_0, ((M-4)/2)_1, (((2k+1)M-4)/4)_1)$ (omit if $M = 8$),
$(0_0, ((3M-4)/4)_1, ((2k+1)M/4)_1)$, and $(0_0, ((3M-8)/8)_1, (((4k+1)M-16)/8)_1)$.

Case 3. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 3 \pmod{12}$. Consider the following collection of blocks:

$(0_1, (kM/3)_1, (2kM/3)_1)$, $(\infty, 0_1, (kM/2)_1)$,
$(0_1, (kM/6-2r)_1, (kM/2-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-4)/12$,
$(0_1, ((kM+6)/6-2r)_1, (kM/3-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-4)/12$,
$(0_1, ((M+1)/3-r)_1, (((3k+3)M-12)/12+r)_1)$ for $r = 1, 2, \ldots, (M+4)/12$,
$(0_0, ((M-8)/12+r)_1, (((5k+2)M-4)/12-r)_1)$ for $r = 1, 2, \ldots, (M-8)/12$,
$(0_0, (((5k+3)M+12)/12-r)_1, (((8k+2)M-16)/12+r)_1)$ for $r = 0, 1, 2, \ldots, (M+4)/12$,
$(0_0, (((5k+2)M-4)/12)_1, (((10k+1)M+4)/12)_1)$
$(0_0, ((5k+3)M/12+r)_1, (((10k+4)M+16)/12-r)_1)$ for $r = 1, 2, \ldots, (M-8)/12$,

$(0_0, (((10k+4)M+16)/12)_1, (((10k+2)M+32)/12)_1)$, $(0_0, (((10k+3)M+12)/12)_1, (((10k+5)M+8)/12)_1)$,
$(0_0, (((10k+4)M+16)/12+r)_1, (((10k+6)M)/12-r)_1)$ for $r=1,2,\ldots,(M-20)/12$.

If $M=32$, also take the two blocks:

$(0_0, (((10k+1)M+16)/12+r)_1, (((10k+1)M+52)/12-r)_1)$ for $r=0,1$.

If $M>32$, instead of the last two blocks, take the blocks:

$(0_0, (((10k+3)M+12)/12-r)_1, (((10k+1)M+16)/12+r)_1)$ for $r=1,2,\ldots,(M-32)/24$,
$(0_0, (((20k+5)M+56)/24-r)_1, (((20k+3)M+48)/24+r)_1)$ for $r=1,2,\ldots,(M-32)/24$,
$(0_0, (((10k+1)M+16)/12)_1, (((10k+2)M+20)/12)_1)$, $(0_0, (((20k+3)M+24)/24)_1, (((20k+3)M+48)/24)_1)$.

Case 4. Suppose that $M\equiv 8 \pmod{24}$ and $k\equiv 5 \pmod{12}$. Consider the following collection of blocks:

$(0_1, (((k-1)M+10)/6+r)_1, (((k-1)M+7)/3-r)_1)$ for $r=1,2,\ldots,((k-1)M-8)/24$,
$(0_1, ((5(k-1)M+56)/24+r)_1, ((7(k-1)M+64)/24-r)_1)$ for $r=1,2,\ldots,((k-1)M-32)/24$,
$(0_1, ((5(k-1)M+56)/24)_1, ((3(k-1)M+24)/8)_1)$, $(0_1, (((k-1)M+8)/4)_1, ((5(k-1)M+44)/12)_1)$,
$(0_1, (((k-1)M+12)/4)_1, ((5(k-1)M+32)/12)_1)$, $(0_1, (((k-1)M+7)/3)_1, ((5(k-1)M+20)/12)_1)$,
$(0_1, (((k-1)M+7)/3+r)_1, (((k-1)M+4)/2-r)_1)$ for $r=1,2,\ldots,((k-1)M-8)/24$,
$(0_1, ((3(k-1)M+24)/8+r)_1, ((11(k-1)M+56)/24-r)_1)$ for $r=1,2,\ldots,((k-1)M-56)/24$,
$(0_0, ((M-2)/2+r)_1, (((k+1)M-6)/2-r)_1)$ for $r=1,2,\ldots,(M-8)/4$,
$(0_0, (r-1)_1, ((kM-4)/2-r)_1)$ for $r=1,2,\ldots,(M-16)/8$,
$(0_0, (M/8+r)_1, (((4k-1)M-8)/8-r)_1)$ for $r=1,2,\ldots,(M-16)/8$,
$(0_0, ((M-16)/8)_1, (((4k-1)M-8)/8))_1$, $(0_0, ((M-8)/8)_1, (M/8)_1)$,
$(0_0, ((M-4)/4)_1, ((kM-4)/2)_1)$, $(0_0, (M/4)_1, (((2k+1)M-8)/4)_1)$,
$(0_0, ((M-2)/2)_1, (((k+1)M-4)/2)_1)$, $(0_0, (M-3)_1, (M-1)_1)$, and $(\infty, 0_1, (kM/2)_1)$.

Case 5. Suppose that $M\equiv 8 \pmod{24}$ and $k\equiv 6 \pmod{12}$. Consider the following collection of blocks:

$(0_1, (kM/3)_1, (2kM/3)_1)$, $(0_1, ((kM-6)/6)_1, ((kM-3)/3)_1)$, $(\infty, 0_1, (kM/2)_1)$,
$(0_1, (((k+1)M-2)/6+r)_1, ((kM-3)/3-r)_1)$ for $r=1,2,\ldots((k-1)M-16)/12$,
$(0_1, (kM/3+r)_1, (((3k-1)M+2)/6-r)_1)$ for $r=1,2,\ldots,((k-1)M-4)/12$.

If $M=8$, also take these four blocks:

$(0_0, 1_1, (((3k+1)M+4)/12)_1)$, $(0_0, (((3k+1)M-8)/12)_1, (((9k+1)M-20)/12)_1)$,
$(0_0, (((9k+1)M-20)/12)_1, (((11k+1)M-44)/12)_1)$, $(0_0, (((3k+1)M+16)/12)_1, (((5k+1)M+28)/12)_1)$.

If $M = 32$, instead of the last four blocks, take these blocks:

$(0_0, ((-kM+4)/4)_1, ((M+4)/12)_1)$,
$(0_0, ((M+10)/6-r)_1, ((kM-6)/6+r)_1)$ for $r = 1, 2, \ldots, (M+4)/12$,
$(0_0, ((M+7)/3-r)_1, (((3k+1)M+4)/6+r)_1)$ for $r = 1, 2, \ldots, (M+4)/12$,
$(0_0, (((-3k+1)M+16)/12-r)_1, (((3k-1)M+8)/12+r)_1)$ for $r = 1, 2, \ldots, (M-8)/12$,
$(0_0, ((9M+48)/24-r)_1, (((4k+9)M+24)/24+r)_1)$ for $r = 1, 2, \ldots, (M-8)/24$,
$(0_0, (((-3k+1)M+16)/12)_1, (((-k+3)M+12/12)_1)$, $(0_0, ((M-1)_1, (((k+5)M+8)/6)_1)$,
$(0_0, (((-3k+1)M+40)/12)_1, (((-k+3)M+24)/12)_1)$, $(0_0, (M-3)_1, (((k+5)M-16)/6)_1)$,
$(0_0, (((-3k+1)M+28)/12)_1, (((-k+1)M+52)/12)_1)$, $(0_0, (M-5)_1, (((k+6)M-12)/6)_1)$.

If $M > 32$, instead of the last two collections of blocks, take these blocks:

$(0_0, ((-kM+4)/4)_1, ((M+4)/12)_1)$,
$(0_0, ((M+10)/6-r)_1, ((kM-6)/6+r)_1)$ for $r = 1, 2, \ldots, (M+4)/12$,
$(0_0, ((M+7)/3-r)_1, (((3k+1)M+4)/6+r)_1)$ for $r = 1, 2, \ldots, (M+4)/12$,
$(0_0, (((-3k+1)M+16)/12-r)_1, (((3k-1)M+8)/12+r)_1)$ for $r = 1, 2, \ldots, (M-8)/12$,
$(0_0, ((9M+48)/24-r)_1, (((4k+9)M+24)/24+r)_1)$ for $r = 1, 2, \ldots, (M-8)/24$,
$(0_0, (((-3k+1)M+4)/12+r)_1, (((-5k+3)M+24)/12-r)_1)$ for $r = 1, 2, \ldots, (M+16)/24$,
$(0_0, (((-6k+3)M+24)/24+r)_1, (((-2k+7)M+64)/24-r)_1)$ for $r = 1, 2, \ldots, (M+16)/24$,
$(0_0, (((-6k+4)M+40)/24)_1, (((-2k+8)M+56)/24-r)_1)$ for $r = 1, 2, \ldots, (M-32)/24$,
$(0_0, (((-2k+7)M+64)/24)_1, (((2k+9)M+96)/24)_1)$, $(0_0, (((2k+9)M+72)/24)_1, (((6k+11)M+8)/24)_1)$,
$(0_0, (((2k+9)M+48)/24)_1, (((6k+11)M+32)/24)_1)$, $(0_0, (((-2k+8)M+56)/24)_1, (((2k+8)M+104)/24)_1)$,
$(0_0, (((k+4)M+40)/12)_1, (((-k+5)M+20)/12)_1)$,
$(0_0, (((6k+11)M+32)/24+r)_1, (((10k+11)M+8)/24-r)_1)$ for $r = 1, 2, \ldots, (M-56)/24$,
$(0_0, (((2k+9)M+48)/24-r)_1, (((6k+9)M+96)/24+r)_1)$ for $r = 1, 2, \ldots, (M-80)/24$.

Case 6. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 9 \pmod{12}$. Consider the following collection of blocks:

$(0_1, (((k+1)M-8)/6+r)_1, (kM/3-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-4)/12$,
$(0_1, (((2k+1)M-2)/6+r)_1, (kM/2-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-4)/12$,

$(0_1, (-1+r)_1, ((kM+6)/6-r)_1)$ for $r = 1, 2, \ldots, (M+4)/12$,
$(0_0, ((M+4)/12)_1, (((3k+2)M-4)/12)_1)$, $(0_0, (((-4k+5)M+8)/24)_1, ((4k+9)M/24)_1)$,
$(0_0, ((M+4)/12+r)_1, (((2k+1)M+16)/12-r)_1)$ for $r = 1, 2, \ldots, (M+4)/12$,
$(0_0, ((2M+8)/12+r)_1, (((-k+1)M+4)/6-r)_1)$ for $r = 1, 2, \ldots, (M-8)/12$,
$(0_0, (3M/12+r)_1, (((4k+5)M-4)/12-r)_1)$ for $r = 1, 2, \ldots, (M-8)/24$,
$(0_0, ((7M-8)/24+r)_1, (((-4k+7)M+16)/24-r)_1)$ for $r = 1, 2, \ldots, (M-8)/12$,
$(0_0, (((-k+1)M-2)/6+r)_1, (((k+2)M+2)/6-r)_1)$ for $r = 1, 2, \ldots, (M-8)/24$,
$(0_0, ((4k+9)M/24-r)_1, ((12k+9)M/24+r)_1)$ for $r = 1, 2, \ldots, (M-8)/24$,
$(0_0, (((12k+10)M-8)/24+r)_1, ((5k+3)M/6-r)_1)$ for $r = 1, 2, \ldots, (M-8)/24$, and
$(\infty, 0_1, (kM/2)_1)$.

Case 7. Suppose that $M \equiv 8 \pmod{24}$ and $k \equiv 11 \pmod{12}$. Consider the following collection of blocks:

$(0_1, (((k-1)M+4)/6+r)_1, (((2k-2)M+2)/6-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-32)/24$,
$(0_1, ((5(k-1)M+32)/24+r)_1, ((7(k-1)M+40)/24-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-32)/24$,
$(0_1, ((8(k-1)M+32)/24+r)_1, (12(k-1)M/24-r)_1)$ for $r = 1, 2, \ldots, ((k-1)M-20)/12$,
$(0_1, ((5(k-1)M+8)/24)_1, ((5(k-1)M+32)/24)_1)$, $(0_1, ((6(k-1)M+24)/24)_1, ((8(k-1)M+32)/24)_1)$,
$(0_1, ((6(k-1)M+48)/24)_1, ((10(k-1)M+16)/24)_1)$, $(0_1, ((8(k-1)M+8)/24)_1, ((k-1)M/2)_1)$,
$(0_0, (M/2+r)_1, ((k+1)M/2-r)_1)$ for $r = 1, 2, \ldots, (M-2)/6$,
$(0_0, ((2M-4)/6+r)_1, (((3k+2)M+2)/6-r)_1)$ for $r = 1, 2, \ldots, (M-2)/6-1$,
$(0_0, 0_1, (((k-1)M+4)/6)_1)$, $(0_0, ((2M-4)/6)_1, ((3kM-6)/6)_1)$, $(0_0, ((2M-10)/6)_1, (3kM/6)_1)$,
$(0_0, ((M+4)/12)_1, (((6k-3)M-24)/12)_1)$, $(0_0, (((3k-2)M+4)/6)_1, (((12k-9)M-12)/12)_1)$,
$(0_0, (((24k-17)M-56)/24)_1, (((36k-29)M-32)/24)_1)$,
$(0_0, (((3k-2)M+4)/6+r)_1, (((6k-4)M-10)/6-r)_1)$ for $r = 0, 1, 2, \ldots, (M-8)/24-1$,
$(0_0, (((6k-3)M-24)/12-r)_1, (((12k-9)M-12)/12+r)_1)$ for $r = 1, 2, \ldots, (M-8)/24-2$
$(0_0, (r)_1, (((3k-2)M+4)/6-r)_1)$ for $r = 1, 2, \ldots, (M-8)/12$, and $(\infty, 0_1, (kM/2)_1)$.

In each case, the given collection of blocks, along with a collection of base blocks for a 1-rotational $STS(M+1)$ on the point set $\{\infty\} \bigcup \mathbb{Z}_M \times \{0\}$ under the automorphism $(\infty)(0_0, 1_0, \ldots, (M-1)_0)$, forms a collection of base blocks for a STS of the desired type. □

Lemmas 1–5 combine to give us necessary and sufficient conditions for the existence of the desired type of STS.

Theorem 3 *A $STS(v)$ admitting an automorphism of type $[\pi] = [1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_1 = \pi_M = \pi_N = 1$, $M < N$, exists if and only if $M \equiv 2$ or 8 (mod 24) and $N = kM$ where*

1. *if $M \equiv 2$ (mod 24) then $k \equiv 2, 3, 6$ or 11 (mod 12),*
2. *if $M \equiv 8$ (mod 24) then $k \equiv 0$ or 2 (mod 3).*

4 Conjectures on the Remaining Tricyclic Steiner Triple Systems

In this section, we consider tricyclic STSs admitting automorphisms of type $[\pi] = [0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_M = \pi_N = \pi_P = 1$, $v = M + N + P$, and $M < N < P$. In our discussion, we will let the point set of such a system be $\mathbb{Z}_M \times \{0\} \bigcup \mathbb{Z}_N \times \{1\} \bigcup \mathbb{Z}_P \times \{2\}$ and let the automorphism be $\pi = (0_0, 1_0, \ldots, (M-1)_0)(0_1, 1_1, \ldots, (N-1)_1)(0_2, 1_2, \ldots, (P-1)_2)$. As in the proof of Theorem 2, by considering π^M, we see that the $STS(v)$ contains a cyclic subsystem of order M and so $M \equiv 0$ or 1 (mod 6), $M \neq 9$. We will now argue that these types of tricyclic STSs fall into three categories. First, if $M = 1$, then the system has a 1-rotational subsystem and necessary and sufficient conditions for such a system were given in Sect. 3. So we now assume $M > 1$.

If $M|N$, say $N = kM$, then by considering π^N, we see that the system has a bicyclic subsystem. Now consider edges of the form (b_1, c_2). Some of the edges of this type can be in blocks of the form (a_0, b_1, c_2). However, there are more edges of the form (b_1, c_2) (namely NP) than there are edges of the form (a_0, c_2) (namely MP). Therefore blocks of the form (a_0, b_1, c_2) cannot contain all of the edges of the form (b_1, c_2). Hence there must be some blocks of (a_1, b_2, c_2). Since π^P fixes edges of the form (b_2, c_2), then π^P must also fix vertex a_1 and so $N \mid P$. We therefore have the following.

Conjecture 1 A $STS(v)$ admitting an automorphism of type $[\pi] = [0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_M = \pi_N = \pi_P = 1$, where $M < N < P$ and $M|N$, exists if and only if

1. $M \equiv 1$ or 3 (mod 6), $M \neq 9$,
2. $M + N \equiv 1$ or 3 (mod 6),
3. $N \mid P$, and
4. $v = M + N + P \equiv 1$ or 3 (mod 6).

Notice that we have hypothesized the condition $M \mid N$ in Conjecture 1. This insures that the tricyclic STS has a bicyclic subsystem. However, we now show by example that this condition is not necessary. Consider vertex set $\{0_0, 1_0, 2_0, 0_1, 1_1, \ldots, 6_1, 0_2, 1_2, \ldots, 20_2\}$ and the collection of blocks: $(0_0, 1_0, 2_0)$, $(0_1, 1_1, 3_1)$, $(0_0, 0_1, 0_2)$, $(0_0, 1_2, 2_2)$, $(0_1, 1_2, 3_2)$, $(0_1, 2_2, 11_2)$, $(0_1, 6_2, 12_2)$, $(0_2, 4_2, 11_2)$, $(0_2, 3_2, 8_2)$. These blocks form a collection of base blocks for a tricyclic STS of order

31 under the permutation $\pi = (0_0, 1_0, 2_0)(0_1, 1_1, \ldots, 6_1)\ (0_2, 1_2, \ldots, 20_2)$. In the notation of this section, we have $M = 3$, $N = 7$, and $P = 21$, so we see that M does not divide N. So the hypothesis $M \mid N$ is not, in general, necessary.

If M does not divide N, then π^N fixes $\{0_1, 1_1, \ldots, (N-1)_1\}$ only, and as in Theorem 2, $N \equiv 1$ or 3 (mod 6), $N \neq 9$. Since such a system must have a cyclic subsystem on $\{0_0, 1_0, \ldots, (M-1)_0\}$ and a cyclic subsystem on $\{0_1, 1_1, \ldots, (N-1)_1\}$, then every edge of the form (a_0, b_1) must be in a block of the form (a_0, b_1, c_2). Since $\pi^{\mathrm{lcm}(M,N)}$ fixes both a_0 and b_1, then it must fix c_2 and so $P \mid \mathrm{lcm}(M, N)$. As described above, there must be some block of the form (a_1, b_2, c_2), π^P fixes edge (b_2, c_2), so π^P must fix a_1 and $N \mid P$. Next, consider edges of the form (a_0, c_2). Similar to the argument above, there must be some block of the form (a_0, b_2, c_2) and by considering π^M we see that $M \mid P$. Since both $M \mid P$ and $N \mid P$, we have that $P \geq \mathrm{lcm}(M, N)$. Hence $P = \mathrm{lcm}(M, N)$. We therefore have the following.

Conjecture 2 A $STS(v)$ admitting an automorphism of type $[\pi] = [0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $\pi_M = \pi_N = \pi_P = 1$, where $M < N < P$ and M does not divide N, exists if and only if

1. $M \equiv 1$ or 3 (mod 6), $M \neq 9$,
2. $N \equiv 1$ or 3 (mod 6), $N \neq 9$,
3. $P = \mathrm{lcm}(M, N)$, and
4. $v = M + N + P \equiv 1$ or 3 (mod 6).

Therefore, we see that the tricyclic STSs of this section fall into three disjoint categories: (1) those with 1-rotational subsystems (as described in Sect. 3), (2) those with bicyclic subsystems (as described in Conjecture 1), and (3) those without bicyclic subsystems (as described in Conjecture 2). We conjecture that the tricyclic STSs of Sect. 2, combined with these STSs, will completely classify tricyclic Steiner triple systems.

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