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4-CYCLE COVERINGS OF THE COMPLETE GRAPH WITH A HOLE

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Abstract: Let $K(v,w)$ denote the complete graph on v vertices with a hole of size w (i.e., $K(v,w) = K_v \setminus K_w$). We give necessary and sufficient conditions for the existence of a minimum 4-cycle *covering* of $K(v,w)$.

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1. Introduction

A *decomposition* of a simple graph G into isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, and $\cup_{i=1}^{n} E(g_i) = E(G)$, where $V(G)$ is the vertex set of graph G and $E(G)$ is the edge set of graph G. A related combinatorial structure is a "graph covering." A *minimum covering* of a simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$, $V(g_i) \subset V(G)$, $E(g_i) \subset E(G)$ for all i, $G \subset \cup_{i=1}^{n} g_i$, and $|\cup_{i=1}^{n} E(g_i) \setminus E(G)|$ is minimum (when

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considering coverings, the graph $\cup_{i=1}^{n} g_i$ may not be simple and $\cup_{i=1}^{n} E(g_i)$ may be a multiset). Coverings of complete graphs have been studied for graph g a 3-cycle [3], a 4-cycle [6], and a 6-cycle [4].

Let $K(v,w)$ denote the complete graph on v vertices with a hole of size w (we assume $w > 0$). Namely, $K(v,w)$ has vertex set $V(K(v,w)) = V_{v-w} \cup V_w$ where $|V_{v-w}| = v - w$ and $|V_w| = w$, and edge set

$$E(K(v,w)) = \{(a,b) \mid a \neq b, \{a,b\} \subset V_{v-w} \cup V_w \text{ and } \{a,b\} \not\subset V_w\}.$$

Necessary and sufficient conditions for the decomposition of $K(v,w)$ into m-cycles are known for $m \in \{3,4,5,6,7,8,10,12,14\}$, see [1, 2, 5].

The purpose of this paper is to give necessary and sufficient conditions for minimum coverings of $K(v,w)$ with copies of a 4-cycle, C_4. Throughout, we denote the 4-cycle with edge set $\{(a,b),(b,c),(c,d),(a,\ d)\}$ as $[a,b,c,d]$ (and analogously for different length cycles).

2. Decompositions

It is rather well known that there is a C_4 decomposition of K_v if and only if $v \equiv 1 \pmod 8$, see [6]. It is also very straightforward to verify that the complete bipartite graph $K_{m,n}$ can be decomposed into copies of C_4 if and only if $m \equiv n \equiv 0 \pmod 2$. A C_4 decomposition of $K(v,w)$ is given in [2]:

Theorem 1. *A C_4 decomposition of $K(v,w)$ exists if and only if $w \equiv 1$ (mod 2) and $v - w \equiv 0$ (mod 8).*

The following lemma is implicit in [6].

Lemma 2. *A decomposition of $K_n \setminus M$, where M is a perfect matching of K_n, into copies of C_4 exists if and only if $n \equiv 0$ (mod 2).*

3. Minimum Coverings

We now give necessary and sufficient conditions for the existence of a minimum covering of $K(v,w)$ with 4-cycles. The parities of w and $v - w$ play a central role and produce a large number of cases in the constructions.

Theorem 3. *A C_4 minimum covering of $K(v,w)$ satisfies the following conditions:*

1. if $v - w \equiv 0$ (mod 2), $v - w > 2$, and $w \equiv 1$ (mod 2), then

$$|E(P)| = \begin{cases} 0 \text{ if } v-w \equiv 0 \pmod 8, \\ 5 \text{ if } v-w \equiv 2 \pmod 8, \\ 2 \text{ if } v-w \equiv 4 \pmod 8, \\ 3 \text{ if } v-w \equiv 6 \pmod 8, \end{cases}$$

2. if $v - w \equiv 0$ (mod 4) and $w \equiv 0$ (mod 2), then $|E(P)| = (v - w)/2$,

3. if $v - w \equiv 2$ (mod 4) and $w \equiv 0$ (mod 2), then $|E(P)| = (v - w)/2 + 2$,

4. if $v-w \equiv 1$ (mod 2), $v-w > 1$, and $w \equiv 0$ (mod 2), then $|E(P)| = w+k$ where k is the minimum nonnegative integer such that $|E(K(v, w))|+|E(P)| \equiv 0$ (mod 4),

5. if $v - w \equiv 1$ (mod 2), $v - w > 1$, $w \equiv 1$ (mod 2), and $v - w \leq w$, then $|E(P)| = w + k$ where k is the minimum nonnegative integer such that $|E(K(v, w))| + |E(P)| \equiv 0$ (mod 4), and

6. if $v-w \equiv 1$ (mod 2), $w \equiv 1$ (mod 2), and $v-w > w$, then $|E(P)| = v/2+k$ where k is the minimum nonnegative integer such that $|E(K(v, w))|+|E(P)| \equiv 0$ (mod 4).

Proof. We consider several cases.

Case 1. Suppose $v - w \equiv 0$ (mod 8) and $w \equiv 1$ (mod 2). Then there exists a decomposition by Theorem 2.1, and $|E(P)| = 0$.

Case 2. Suppose $v - w \equiv 2$ (mod 8) and $w \equiv 1$ (mod 2). First we observe that $v - w > 2$ is necessary since with $v - w = 2$, we see that C_4 is not a subgraph of $K(v, w)$ and so no covering exists. Each vertex of $K(v, w)$ is of even degree, so each vertex in the padding P of a minimum covering will be of even degree. Since $|E(K(v, w)| \equiv 3$ (mod 4), then a padding with one edge would be optimal. However, P cannot have each vertex of even degree and only one edge. So $|E(P)| \geq 5$. Now $K(v, w) = K_{v-w-1} \cup K_{v-w,w-1} \cup (v - w - 2)/2 \times C_4 \cup C_3$ where the vertex set of K_{v-w-1} is $V_{v-w} \setminus \{(v-w)_1\}$, the vertex set of $K_{v-w,w-1}$ has partite sets V_{v-w} and $V_w \setminus \{w_2\}$, $(v - w - 2)/2 \times C_4 = \{[(v - w)_1, (2i - 1)_1, w_2, (2i)_1] \mid i = 1, 2, \dots, (v-w-2)/2\}$, and $C_3 = [(v-w-1)_1, (v-w)_1, w_2]$. Since K_{v-w-1} and $K_{v-w,w-1}$ can be decomposed into C_4s, then by combining the blocks of these decompositions with $[(v-w)_1, w_2, (v-w-3)_1, (v-w-2)_1]$ and $[(v-w)_1, (v-w-1)_1, w_2, (v-w-4)_1]$, we get a minimum covering of $K(v, w)$ with a padding P where $P = [(v-w-3)_1, (v-w-2)_1, (v-w)_1, (v-w-4)_1, w_2]$ and so $|E(P)| = 5$.

Case 3. Suppose $v-w \equiv 4$ (mod 8) and $w \equiv 1$ (mod 2). Then $|E(K(v, w))| \equiv 2$ (mod 4), and so a covering with $|E(P)| = 2$ would be optimal. Now $K(v, w) =$

$K_{v-w-3} \cup K_{v-w,w-1} \cup (v-w-3) \times C_4 \cup 2 \times C_3$ where the vertex set of K_{v-w-3} is $\{1_1, 2_1, \ldots, (v-w-3)_1\}$, the vertex set of $K_{v-w,w-1}$ has partite sets V_{v-w} and $V_w \setminus \{w_2\}$, $(v-w-2) \times C_4 = \{[(v-w)_1, (2i-1)_1, w_2, (2i)_1], [(v-w-1)_1, (2i-1)_1, (v-w-2)_1, (2i)_1] \mid i = 1, 2, \ldots, (v-w-4)/2\} \cup \{[(v-w-1)_1, (v-w-2)_1, (v-w-3)_1, w_2]\}$, and $2 \times C_3 = [(v-w)_1, (v-w-1)_1, (v-w-3)_1] \cup [(v-w)_1, (v-w-2)_1, w_2]$. Since K_{v-w-3} and $K_{v-w,w-1}$ can be decomposed into C_4s, then by combining the blocks of these decompositions with $[(v-w)_1, (v-w-1)_1, (v-w-3)_1, w_2]$ and $[(v-w)_1, (v-w-3)_1, w_2, (v-w-2)_1]$, we get a minimum covering of $K(v,w)$ with a padding P where $P = 2 \times ((v-w-3)_1, w_2)$ and $|E(P)| = 2$.

Case 4. Suppose $v-w \equiv 6 \pmod 8$ and $w \equiv 1 \pmod 2$. Then $|E(K(v,w))| \equiv 1 \pmod 4$ and so a covering with $|E(P)| = 3$ would be optimal. Now $K(v,w) = K_{v-w-5} \cup K_{v-w,w-1} \cup (3(v-w-6)/2+4) \times C_4 \cup C_5$ where the vertex set of K_{v-w-5} is $\{1_1, 2_1, \ldots, (v-w-5)_1\}$, the vertex set of $K_{v-w,w-1}$ has partite sets V_{v-w} and $V_w \setminus \{w_2\}$, $(3(v-w-6)/2+4) \times C_4 = \{[(v-w)_1, (2i-1)_1, w_2, (2i)_1], [(v-w-1)_1, (2i-1)_1, (v-w-2)_1, (2i)_1], [(v-w-3)_1, (2i-1)_1, (v-w-4)_1, (2i)_1] \mid i = 1, 2, \ldots, (v-w-6)/2\} \cup \{[w_2, (v-w-2)_1, (v-w-3)_1, (v-w)_1], [w_2, (v-w-3)_1, (v-w-1)_1, (v-w-4)_1], [(v-w-5)_1, (v-w-3)_1, (v-w-4)_1, (v-w)_1], [(v-w-5)_1, (v-w-1)_1, (v-w)_1, (v-w-2)_1]\}$, and $C_5 = [(v-w-5)_1, w_2, (v-w-1)_1, (v-w-2)_1, (v-w-4)_1]$. Since K_{v-w-5} and $K_{v-w,w-1}$ can be decomposed into C_4s, then by combining the blocks of the decompositions with $[(v-w-5)_1, (v-w-1)_1, (v-w-2)_1, (v-w-4)_1]$ and $[w_2, (v-w-1)_1, (v-w-6)_1, (v-w-5)_1]$, we get a minimum covering of $K(v,w)$ with padding P where $P = [(v-w-1)_1, (v-w-6)_1, (v-w-5)_1]$ and so $|E(P)| = 3$.

Case 5. Suppose $v-w \equiv 0 \pmod 4$ and $w \equiv 0 \pmod 2$. Since each vertex of V_{v-w} is of odd degree, then in the padding P of an optimal covering, each vertex must be of odd degree. Hence a covering with $|E(P)| = (v-w)/2$ would be optimal. Now $K(v,w) = (K_{v-w} \setminus M) \cup K_{v-w,w} \cup M$ where the vertex set of $K_{v-w} \setminus M$ is V_{v-w}, the vertex set of $K_{v-w,w}$ has partite sets V_{v-w} and V_w, and M is a perfect matching of the K_{v-w}. Say $E(M) = \{l_1, l_2, \ldots, l_{(v-w)/2}\}$, where $E(l_i) = \{((2i-1)_1, (2i)_1) \mid i = 1, 2, \ldots, (v-w)/2\}$. Since $K_{v-w} \setminus M$ can be decomposed into C_4s by Lemma 2 and $K_{v-w,w}$ can be decomposed into C_4s, then by combining the blocks of the decompositions with the 4-cycles $\{[(4i-3)_1, (4i-2)_1, (4i-1)_1, (4i)_1] \mid i = 1, 2, \ldots, (v-w)/4\}$ we get a minimum covering of $K(v,w)$ where P is a matching of V_{v-w} with edge set $E(P) = \{((4i-2)_1, (4i-1)_1), ((4i-3)_1, (4i)_1) \mid i = 1, 2, \ldots, (v-w)/4\}$, and so $|E(P)| = (v-w)/2$.

Case 6. Suppose $v-w \equiv 2 \pmod 4$ and $w \equiv 0 \pmod 2$. As in Case 5, a covering satisfying $|E(P)| = (v-w)/2$ would be optimal. However, $|E(K(v,w))| + (v-w)/2 \equiv 2 \pmod 4$. Hence, a covering satisfying $|E(P)| = (v-w)/2 + 2$ would be optimal. Now $K(v,w) = (K_{v-w} \setminus M) \cup K_{v-w,w} \cup M$ where the vertex set of $K_{v-w} \setminus M$ is V_{v-w}, the vertex set of $K_{v-w,w}$ has partite sets V_{v-w} and V_w, and M is a perfect matching of the K_{v-w}. Say $E(M) = \{l_1, l_2, \ldots, l_{(v-w)/2}\}$, where $E(l_i) = \{((2i-1)_1, (2i)_1) \mid i = 1, 2, \ldots, (v-w)/2\}$. Since $K_{v-w} \setminus M$ can be decomposed into C_4s by Lemma 2 and $K_{v-w,w}$ can be decomposed into C_4s, then by combining the blocks of the decompositions with the 4-cycles $\{[(4i-3)_1, (4i-2)_1, (4i-1)_1, (4i)_1] \mid i = 1, 2, \ldots, (v-w-2)/4\} \cup \{[(v-w-1)_1, (v-w)_1, (v-w-3)_1, (v-w-2)_1]\}$ we get a minimum covering of $K(v,w)$ where $E(P) = \{((4i-2)_1, (4i-1)_1), ((4i-3)_1, (4i)_1) \mid i = 1, 2, \ldots, (v-w-2)/4\} \cup \{((v-w-3)_1, (v-w)_1), ((v-w-3)_1, (v-w-2)_1), ((v-w-2)_1, (v-w-1)_1)\}$, and so $|E(P)| = (v-w)/2 + 2$

Case 7. Suppose $v-w \equiv 1 \pmod 8$ and $w \equiv 0 \pmod 2$. First we observe that $v-w > 1$ is necessary since with $v-w = 1$, we see that C_4 is not a subgraph of $K(v,w) = S_{v-1}$ and so no covering exists. Each vertex of V_w is of odd degree, and so in an optimal covering with padding P we would have $|E(P)| \geq w$ (recall that edges *within* vertex set V_w are not allowed). Now $K(v,w) = K_{v-w} \cup K_{v-w-1,w} \cup S_w$ where the vertex set of K_{v-w} is V_{v-w}, the vertex set of $K_{v-w-1,w}$ has partite sets $V_{v-w} \setminus \{(v-w)_1\}$ and V_w, and S_w is a star with edge set $\{((v-w)_1, i_2) \mid i = 1, 2, \ldots, w\}$. Since K_{v-w} and $K_{v-w-1,w}$ can be decomposed into C_4s, then by combining the blocks of the decompositions with the 4-cycles $\{[(v-w)_1, (2i-1)_2, (v-w-1)_1, (2i)_2] \mid i = 1, 2, \ldots, w/2\}$ we get a minimum covering of $K(v,w)$ where $P = S_w$ where $E(P) = \{((v-w-1)_1, i_2) \mid i = 1, 2, \ldots, w\}$, and so $|E(P)| = w$.

Case 8. Suppose $v-w \equiv 3 \pmod 8$ and $w \equiv 0 \pmod 2$. As in Case 7, an optimal covering with padding P satisfies $|E(P)| \geq w$. Since $|E(K(v,w))| + w \equiv 3 \pmod 4$, then we need $|E(P)| \geq w+1$. Now $K(v,w) = K_{v-w-2} \cup K_{v-w-3,2} \cup K_{v-w-3,w} \cup K_{2,w} \cup S_w \cup C_3$ where the vertex set of K_{v-w-2} is $\{1_1, 2_1, \ldots, (v-w-2)_1\}$, the vertex set of $K_{v-w-3,2}$ has partite sets $\{1_1, 2_1, \ldots, (v-w-3)_1\}$ and $\{(v-w-1)_1, (v-w)_1\}$, the vertex set of $K_{v-w-3,w}$ has partite sets $\{1_1, 2_1, \ldots, (v-w-3)_1\}$ and V_w, the vertex set of $K_{2,w}$ has partite sets $\{(v-w-1)_1, (v-w)_1\}$ and V_w, the edge set of S_w is $\{(v-w-2)_1, i_2) \mid i = 1, 2, \ldots, w\}$, and the edge set of C_3 is $\{((v-w-2)_1, (v-w-1)_1), ((v-w-1)_1, (v-w)_1), ((v-w-2)_1, (v-w)_1)\}$. Since K_{v-w-2}, $K_{v-w-3,2}$, and $K_{v-w-3,w}$ can be decomposed into C_4s, then by combining the blocks of the decompositions with the 4-cycles $\{[(v-w-2)_1, (v-w-1)_1, (v-w)_1, 1_2], [(v-w-2)_1, (v-w)_1, (v-w-1)_1, 2_2]\} \cup \{[(v-w-2)_1, (2i+1)_2, (v-w)_1, (2i+2)_2] \mid$

$i = 1, 2, \ldots, w/2 - 1\}$ we get a minimum covering of $K(v, w)$ where $P = S_{w+1}$ with $E(P) = \{((v-w-1)_1, (v-w)_1)\} \cup \{((v-w)_1, i_2) \mid i = 1, 2, \ldots, w\}$ and so $|E(P)| = w + 1$.

Case 9. Suppose $v - w \equiv 5 \pmod 8$ and $w \equiv 0 \pmod 2$. As in Case 7, an optimal covering with padding P satisfies $|E(P)| \geq w$. Since $|E(K(v,w)| + w \equiv 2 \pmod 4$, then we need $|E(P)| \geq w + 2$. Now $K(v, w) = K_{v-w-4} \cup K_{v-w-5,4} \cup K_{v-w-3,w} \cup K_{2,w-2} \cup 3 \times C_4 \cup S_{w-2} \cup P_4$ where the vertex set of K_{v-w-4} is $\{1_1, 2_1, \ldots, (v-w-4)_1\}$, the vertex set of $K_{v-w-5,4}$ has partite sets $\{1_1, 2_1, \ldots, (v-w-5)_1\}$ and $\{(v-w-3)_1, (v-w-2)_1, (v-w-1)_1, (v-w)_1\}$, the vertex set of $K_{v-w-3,w}$ has partite sets $\{1_1, 2_1, \ldots, (v-w-3)_1\}$ and V_w, the vertex set of $K_{2,w-2}$ has partite sets $\{(v-w)_1, (v-w-2)_1\}$ and $\{1_2, 2_2, \ldots, , (w-2)_2\}$, the edge set of S_{w-2} is $\{((v-w-1)_1, i_2) \mid i = 1, 2, \ldots w-2\}$, the edge set of P_4 is $\{((w-1)_2, (v-w-2)_1), ((v-w-2)_1, (v-w-1)_1), ((v-w-1)_1, (v-w)_1), ((v-w)_1, w_2)\}$, and: $3 \times C_4 = \{[(v-w-4)_1, (v-w-3)_1, (v-w)_1, (v-w-2)_1], [(v-w-4)_1, (v-w-1)_1, (w-1)_2, (v-w)_1], [(v-w-3)_1, (v-w-2)_1, w_2, (v-w-1)_1]\}$. Since K_{v-w-4}, $K_{v-w-5,4}$, and K_{v-w-3} can be decomposed into copies of C_4, then by combining the blocks of the decompositions with the 4-cycles $\{[(v-w)_1, (2i-1)_2, (v-w-1)_1, (2i)_2)] \mid i = 1, 2, \ldots, (w-2)/2\} \cup \{[(v-w-2)_1, (v-w)_1, w_2, (v-w-1)_1], [(v-w-2)_1, (v-w)_1, (v-w-1)_1, (w-1)_2]\}$ we get a minimum covering of $K(v, w)$ where $P = S_w$ with $E(P) = \{(v-w)_1, i_2) \mid i = 1, 2, \ldots, w\} \cup 2 \times \{((v-w)_1, (v-w-2)_1)\}$, and so $|E(P)| = w + 2$.

Case 10. Suppose $v - w \equiv 7 \pmod 8$ and $w \equiv 0 \pmod 2$. As in Case 7, an optimal covering with padding P satisfies $|E(P)| \geq w$. Since $|E(K(v,w)| + w \equiv 1 \pmod 4$, then we need $|E(P)| \geq w + 3$. Now $K(v, w) = K_{v-w-6} \cup K_{v-w-7,6} \cup K_{v-w-7,w} \cup K_{6,w-2} \cup K_{4,2} \cup 6 \times C_4 \cup S_{w-2} \cup P_3$ where the vertex set of K_{v-w-6} is $\{1_1, 2_1, \ldots, (v-w-6)_1\}$, the vertex set of $K_{v-w-7,6}$ has partite sets $\{1_1, 2_1, \ldots, (v-w-7)_1\}$ and $\{(v-w-5)_1, (v-w-4)_1, \ldots, (v-w)_1\}$, the vertex set of $K_{v-w-7,w}$ has partite sets $\{1_1, 2_1, \ldots, (v-w-7)_1\}$ and V_w, the vertex set of $K_{6,w-2}$ has partite sets $\{(v-w-5)_1, (v-w-4)_1, \ldots, (v-w)_1\}$ and $\{1_2, 2_2, \ldots, (w-2)_2\}$, the vertex set of $K_{4,2}$ has partite sets $\{(v-w-3)_1, (v-w-2)_1, (v-w-1)_1, (v-w)_1\}$ and $\{(w-1)_2, w_2\}$, the edge set of S_{w-2} is $\{((v-w-6)_1, i_2) \mid i = 1, 2, \ldots, w-2\}$, the edge set of P_3 is $\{((v-w-5)_1, (w-1)_2), ((v-w-5)_1, (v-w-4)_1), ((v-w-4)_1, w_2)\}$, and: $6 \times C_4 = \{[(v-w-6)_1, (v-w-3)_1, (v-w-4)_1, (w-1)_2], [(v-w-6)_1, (v-w-2)_1, (v-w-5)_1, w_2], [(v-w-6)_1, (v-w-5)_1, (v-w-3)_1, (v-w)_1], [(v-w-6)_1, (v-w-4)_1, (v-w-2)_1, (v-w-1)_1][(v-w-3)_1, (v-w-2)_1, (v-w)_1, (v-w-1)_1], [(v-w-5)_1, (v-w-1)_1, (v-w-4)_1, (v-w)_1]\}$. Since K_{v-w-6}, $K_{v-w-7,6}$, $K_{v-w-7,w}$, $K_{6,w-2}$, and $K_{4,2}$ can be decomposed into copies of C_4, then by combining the blocks of the decompositions with the 4-cycles

$\{[(v-w)_1,(2i-1)_2,(v-w-6)_1,(2i)_2] \mid i=1,2,\ldots,(w-2)/2\} \cup \{[(v-w-5)_1,(v-w-4)_1,(v-w-3)_1,(w-1)_2],[(v-w-4)_1,(v-w-3)_1,(v-w-2)_1,w_2]\}$ we get a minimum covering of $K(v,w)$ where $P = S_{w-2} \cup P_3 \cup 2 \times K_2$ with $E(S_{w-2}) = \{((v-w)_1, i_2) \mid i = 1,2,\ldots,w-2\}$, $E(P_3) = \{((w-1)_2,(v-w-3)_1),((v-w-3)_1,(v-w-2)_1),((v-w-2)_1,w_2)\}$, and $E(2 \times K_2) = 2 \times \{((v-w-4)_1,(v-w-3)_1)\}$, and so $|E(P)| = w+3$.

Case 11. Suppose $v-w \equiv 1 \pmod 8$ and $w \equiv 1 \pmod 2$, where $v-w \leq w$. First we observe that $v-w > 1$ is necessary since with $v-w=1$, we see that C_4 is not a subgraph of $K(v,w) = S_{v-1}$ and so no covering exists. As in Case 7, an optimal covering with padding P satisfies $|E(P)| \geq w$. Since $|E(K(v,w))| + w \equiv 2 \pmod 4$, then we need $|E(P)| \geq w+2$. Now $K(v,w) = K_{v-w} \cup K_{v-w-1,2w-v} \cup (K_{v-w,v-w} \setminus M) \cup M \cup S_{2w-v}$ where the vertex set of K_{v-w} is V_{v-w}, the vertex set of $K_{v-w-1,2w-v}$ has partite sets $V_{v-w} \setminus \{(v-w)_1\}$ and $\{(v-w+1)_2,(v-w+2)_2,\ldots,w_2\}$, the edge set of $K_{v-w,v-w}$ has partite sets V_{v-w} and $\{1_2,2_2,\ldots,(v-w)_2\}$, M has edge set $\{(i_1,i_2) \mid i=1,2,\ldots,(v-w)\}$, and the edge set of S_{2w-v} is $\{((v-w)_1,(v-w+i)_2) \mid i=1,2,\ldots,2w-v\}$. Since K_{v-w}, $K_{v-w-1,2w-v}$, and $K_{v-w,v-w} \setminus M$ can be decomposed into C_4s, then by combining the blocks of the decompositions with the 4-cycles $\{[(2i-1)_1,(2i-1)_2,(2i)_1,(2i)_2] \mid i=1,2,\ldots,(v-w-1)/2\} \cup \{[(v-w-1)_1,(v-w+2i-1)_2,(v-w)_1,(v-w+2i)] \mid i=1,2,\ldots,(2w-v)/2\} \cup \{(v-w-2)_1,(v-w-1)_1,(v-w)_1,(v-w)_2]\}$ we get a minimum covering of $K(v,w)$ where $P = M' \cup S_{2w-v} \cup P_3$ with $E(M') = \{((2i-1)_1,(2i)_2),((2i)_1,(2i-1)_2) \mid i=1,2,\ldots,(v-w-1)/2\}$, $E(S_{2w-v}) = \{((v-w-1)_1,(v-w+i)_2) \mid i=1,2,\ldots,2w-v\}$, and $E(P_3) = \{((v-w-2)_1,(v-w-1)_1),((v-w-1)_1,(v-w)_1),((v-w-2)_1,(v-w)_2)\}$ and so $|E(P)| = w+2$.

Case 12. Suppose $v-w \equiv 3 \pmod 8$ and $w \equiv 1 \pmod 2$, where $v-w \leq w$. As in Case 7, an optimal covering with padding P satisfies $|E(P)| \geq w$. Since $|E(K(v,w))| + w \equiv 3 \pmod 4$, then we need $|E(P)| \geq w+1$. Now $K(v,w) = K_{v-w-2} \cup (K_{v-w-2,v-w-2} \setminus M) \cup K_{v-w-3,2} \cup C_4 \cup K_{2,v-w-3} \cup K_{v-w-1,2w-v+2} \cup M \cup S_{2w-v+2} \cup K_2$ where the vertex set of K_{v-w-2} is $V_{v-w} \setminus \{(v-w-1)_1,(v-w)_1,\}$, the vertex set of $K_{v-w-2,v-w-2}$ has partite sets $\{1_1,2_1,\ldots,(v-w-2)_1\}$ and $\{1_2,2_2,\ldots,(v-w-2)_2\}$, the edge set of M is $\{(i_1,i_2) \mid i=1,2,\ldots,v-w-2\}$, the vertex set of $K_{v-w-3,2}$ has partite sets $V_{v-w} \setminus \{(v-w-2)_1,(v-w-1)_1,(v-w)_1\}$ and $\{(v-w-1)_1,(v-w)_1\}$, $C_4 = [(v-w-2)_1,(v-w)_1,(v-w-2)_2,(v-w-1)_1]$, the vertex set of $K_{2,v-w-3}$ has partite sets $\{(v-w-1)_1,(v-w)_1\}$ and $\{1_2,2_2,\ldots,(v-w-3)_2\}$, the vertex set of $K_{v-w-1,2w-v+2}$ has partite sets $V_{v-w} \setminus \{(v-w-2)_1\}$ and $\{(v-w-1)_2,(v-w)_2,\ldots,w_2\}$, the edge set of S_{2w-v+2} is $\{(v-w-2)_1,(v-w-2+i)_2 \mid i=1,2,\ldots,2w-v+2\}$, and the edge set of K_2 is $\{((v-w-1)_1,(v-w)_1\}$. Since K_{v-w-2}, $K_{v-w-2,v-w-2} \setminus$

M, $K_{v-w-3,2}$, $K_{2,v-w-3}$, $K_{v-w-1,2w-v+2}$, can each be decomposed into copies of C_4, then by combining the blocks of the decompositions with the 4-cycles $\{[(2i-1)_1,(2i-1)_2,(2i)_1,(2i)_2)] \mid i=1,2,\ldots,(v-w-3)/2\} \cup \{[(v-w)_1,(v-w+2i-3)_2,(v-w-2)_1,(v-w+2i-2)_2] \mid i=1,2,\ldots,(2w-v+2)/2\} \cup \{[(v-w-2)_1,(v-w-1)_1,(v-w)_1,(v-w-2)_2]\}$ we get a minimum covering of $K(v,w)$ where $P=M' \cup S_{2w-v+3} \cup K_2$ with $E(M')=\{((2i-1)_1,(2i)_2)), ((2i)_1,(2i-1)_2) \mid i=1,2,\ldots,(v-w-3)/2\}$, $E(S_{2w-v+3})=\{((v-w)_1,(v-w+i-3)_2) \mid i=1,2,\ldots,2w-v+3\}$, and $E(K_2)=\{(v-w-2)_1,(v-w-1)_1)\}$, and so $|E(P)|=w+1$.

Case 13. Suppose $v-w \equiv 5 \pmod 8$ and $v \equiv 1 \pmod 2$, where $v-w \leq w$. As in Case 7, an optimal covering with padding P satisfies $|E(P)| \geq w$. In this case, we assume the vertex set of $K(v,w)$ is $V(K(v,w)) = V'_{v-w} \cup V'_w$ where $V'_{v-w}=\{0_1,1_1,\ldots,(v-w-1)_1\}$ and $V'_w=\{0_2,1_2,\ldots,(w-1)_2\}$. Consider the following set of 4-cycles (where the vertex labels are reduced modulo $v-w$): $G=\{[j_1,(4i+j)_1,(1+j)_1,(4i-2+j)_1] \mid i=1,2,\ldots,(v-w-5)/8, j=1,2,\ldots,v-w\} \cup \{[(i-1)_1,(i-1)_2,i_1,((v-w-3+2i)/2)_1] \mid i=1,2,\ldots,v-w\}$. Then $K(v,w)=G \cup K_{2,2w-v} \cup K_{2,2w-v} \cup K_{v-w-3,2w-v} \cup (K_{v-w,v-w} \setminus M$ where the vertex set of the first $K_{2,2w-v}$ has partite sets $\{(v-w-2)_1,(v-w-1)_1\}$ and $\{(v-w)_2,(v-w+1)_2,\ldots,(w-1)_2\}$, the vertex set of the second $K_{2,2w-v}$ has partite sets $\{(v-w-3)_1,(v-w-1)_1\}$ and $\{(v-w)_2,(v-w+1)_2,\ldots,(w-1)_2\}$, the vertex set of $K_{v-w-3,2w-v}$ has partite sets $\{0_1,1_1,\ldots,(v-w-4)_1\}$ and $\{(v-w)_2,(v-w+1)_2,\ldots,(w-1)_2\}$, the vertex set of $K_{v-w,v-w}$ has partite sets $\{0_1,1_1,\ldots,(v-w-1)_1\}$ and $\{0_2,1_2,\ldots,(v-w-1)_2\}$, and $E(M)=\{(i_1,i_2) \mid i=0,1,\ldots,v-w-1\}$. Since $K_{2,2w-v}$, $K_{v-w-3,2w-v}$, and $(K_{v-w,v-w} \setminus M)$ can be decomposed into copies of C_4, then there exists a minimum covering of $K(v,w)$ with padding $P=M' \cup S_{2w-v}$ with $E(M')=\{(i_1,(i-1)_2) \mid i=1,2,\ldots,v-w-1\} \cup \{(0_1,(w-v-1)_2)\}$ and $E(S_{2w-v})=\{((v-w-1)_1,(v-w-1+i)_2) \mid i=1,2,\ldots,2w-v\}$, and so $|E(P)|=w$.

Case 14. Suppose $v-w \equiv 7 \pmod 8$ and $w \equiv 1 \pmod 2$, where $v-w \leq w$. As in Case 7, an optimal covering with padding P satisfies $|E(P)| \geq w$. Since $|E(K(v,w))|+w \equiv 1 \pmod 4$, then we need $|E(P)| \geq w+3$. Now $K(v,w)=K_{v-w-6} \cup K_{v-w-7,6} \cup (K_{v-w-6,v-w-6} \setminus M) \cup K_{v-w-1,2w-v+6} \cup K_{6,v-w-7} \cup 6 \times C_4 \cup M \cup S_{2w-v+6} \cup 3 \times K_2$ where the vertex set of K_{v-w-6} is $\{1_1,2_1,\ldots,(v-w-6)_1\}$, the vertex set of $K_{v-w-7,6}$ has partite sets $\{1_1,2_1,\ldots,(v-w-7)_1\}$ and $\{(v-w-5)_1,(v-w-4)_1,(v-w-3)_1,(v-w-2)_1,(v-w-1)_1,(v-w)_1\}$, $K_{v-w-6,v-w-6}$ has partite sets $\{1_1,2_1,\ldots,(v-w-6)_1\}$ and $\{1_2,2_2,\ldots,(v-w-6)_2\}$, M has edge set $\{(i_1,i_2) \mid i=1,2,\ldots,v-w-6\}$, the vertex set of $K_{v-w-1,2w-v+6}$ has partite sets $V_{v-w} \setminus \{(v-w-6)_1\}$ and $\{(v-w-5)_2,(v-w-4)_2,\ldots,w_2\}$, the vertex set of $K_{6,v-w-7}$ has partite sets $\{(v-w-5)_1,(v-w-4)_1,\ldots,(v-w)_1\}$

and $\{1_2, 2_2, \ldots, (v-w-7)_2\}$, $6 \times C_4 = \{[(v-w-6)_1, (v-w-5)_1, (v-w-6)_2, (v-w-4)_1], [(v-w-6)_1, (v-w-3)_1, (v-w-6)_2, (v-w-2)_1], [(v-w-6)_1, (v-w-1)_1, (v-w-6)_2, (v-w)_1], [(v-w-5)_1, (v-w-4)_1, (v-w-2)_1, (v-w-1)_1], [(v-w-4)_1, (v-w-3)_1, (v-w-1)_1, (v-w)_1], [(v-w-5)_1, (v-w-3)_1, (v-w-2)_1, (v-w)_1]\}$, the edge set of S_{2w-v+6} is $\{((v-w-6)_1, (v-w-6+i)_2) \mid i = 1, 2, \ldots, 2w-v+6\}$, and the edge set of $3 \times K_2$ is $\{((v-w-5)_1, (v-w-2)_1), ((v-w-3)_1, (v-w)_1), ((v-w-4)_1, (v-w-1)_1)\}$. Since K_{v-w-6}, $K_{v-w-7,6}$, $(K_{v-w-6,v-w-6} \setminus M)$, $K_{v-w-1,2w-v+6}$, and $K_{6,v-w-7}$ can be decomposed into copies of C_4, then by combining the blocks of the decompositions with the 4-cycles $\{[(2i-1)_1, (2i-1)_2, (2i)_1, (2i)_2] \mid i = 1, 2, \ldots, (v-w-7)/2\} \cup \{[(v-w)_1, (v-w+2i-7)_2, (v-w-6)_1, (v-w+2i-6)_2] \mid i = 1, 2, \ldots, (2w-v+6)/2\} \cup \{((v-w-6)_1, (v-w-3)_1), ((v-w)_1, (v-w-6)_1)), ((v-w-5)_1, (v-w-4)_1), ((v-w-1)_1, (v-w-2)_2)\}$ we get a minimum covering of $K(v,w)$ where $P = M' \cup S_{2w-v+6} \cup 4 \times K_2$ with $E(M') = \{((2i-1)_1, (2i)_2), ((2i)_1, (2i-1)_2) \mid i = 1, 2, \ldots, (v-w-7)/2\}$, $E(S_{2w-v+6}) = \{((v-w)_1, (v-w+i-6)_2) \mid i = 1, 2, \ldots, 2w-v+6\}$, and $E(4 \times K_2) = \{((v-w-6)_1, (v-w-3)_1), ((v-w-5)_1, (v-w-4)_1), ((v-w-2)_1, (v-w-1)_1), ((v-w)_1, (v-w-6)_2)\}$, and so $|E(P)| = w+3$.

Case 15. Suppose $v \equiv 0 \pmod 4$ and $w \equiv 1 \pmod 8$, where $v - w \geq w$. Since each vertex of $K(v,w)$ is of odd degree, then in an optimal covering of $K(v,w)$ with padding P, each vertex of P must be of odd degree. Therefore $|E(P)| \geq v/2$. Now $K(v,w) = K_w \cup K_{w-1,v-2w} \cup K_{v-2w,w-1} \cup (v-2w)/2 \times C_4 \cup (K_{w,w} \setminus M_1) \cup (K_{v-2w} \setminus M_2) \cup M_1 \cup M_2$ where the vertex set of K_w is $\{1_1, 2_1, \ldots, w_1\}$, the vertex set of $K_{w-1,v-2w}$ has partite sets $\{1_1, 2_1, \ldots, (w-1)_1\}$ and $\{(w+1)_1, (w+2)_1, \ldots, (v-w)_1\}$, the vertex set of $K_{v-2w,w-1}$ has partite sets $\{(w+1)_1, (w+2)_1, \ldots, (v-w)_1\}$ and $V_w \setminus \{w_2\}$, $(v-2w)/2 \times C_4 = \{[w_1, (w+2i-1)_1, w_2, (w+2i)_1] \mid i = 1, 2, \ldots, (v-2w)/2\}$, $K_{w,w}$ has partite sets $\{1_1, 2_1, \ldots, w_1\}$ and V_w, the edge set of M_1 is $\{(i_1, i_2) \mid i = 1, 2, \ldots, w\}$, the vertex set of K_{v-2w} is $\{(w+1)_1, (w+2)_1, \ldots, (v-w)_1\}$, and the edge set of M_2 is $\{((w+2i-1)_1, (w+2i)_1) \mid i = 1, 2, \ldots, (v-2w)/2\}$. Since K_w, $K_{w-1,v-2w}$, $K_{v-2w,w-1}$, $(K_{w,w} \setminus M_1)$, and $(K_{v-2w} \setminus M_2)$ can be decomposed into copies of C_4, then by combining the blocks of the decompositions with the 4-cycles $\{[(2i-1)_1, (2i-1)_2, (2i)_1, (2i)_2] \mid i = 1, 2, \ldots, (w-1)/2\} \cup \{[(w+4i-1)_1, (w+4i)_1, (w+4i+1)_1, (w+4i+2)_1] \mid i = 1, 2, \ldots, (v-2w-2)/4\} \cup \{[w_1, (w+1)_1, (w+2)_1, w_2]\}$ we get a minimaum covering of $K(v,w)$ where $P = M'$ with $E(M') = \{((2i-1)_1, (2i)_2), ((2i)_1, (2i-1)_2) \mid i = 1, 2, \ldots, (w-1)/2\} \cup \{(w_1, (w+1)_1), (w+2)_1, w_2)\} \cup \{((w+4i-1)_1, (w+4i+2)_1), ((w+4i)_1, (w+4i+1)_1) \mid i = 1, 2, \ldots, (v-2w-2)/4\}$, and so $|E(P)| = v/2$.

Case 16. Suppose $v \equiv 2 \pmod 4$ and $w \equiv 1 \pmod 8$, where $v - w \geq w$. As in Case 15, an optimal covering with padding P satisfies $|E(P)| \geq v/2$. Since $|E(K(v,w))| + v/2 \equiv 2 \pmod 4$, then we need $|E(P)| \geq v/2 + 2$. Now $K(v,w) = K_w \cup K_{w-1,v-2w} \cup K_{v-2w,w-1} \cup (v-2w)/2 \times C_4 \cup (K_{w,w} \setminus M_1) \cup (K_{v-2w} \setminus M_2) \cup M_1 \cup M_2$, as established in Case 15 (with the vertex sets as given in Case 15). Since K_w, $K_{w-1,v-2w}$, $K_{v-2w,w-1}$, $(K_{w,w} \setminus M_1)$, and $(K_{v-2w} \setminus M_2)$ can be decomposed into copies of C_4, then by combining the blocks of the decompositions with the 4-cycles $\{[(2i-1)_1, (2i-1)_2, (2i)_1, (2i)_2] \mid i = 1, 2, \ldots, (w-1)/2\} \cup \{[(w+4i-1)_1, (w+4i)_1, (w+4i+1)_1, (w+4i+2)_1] \mid = 1, 2, \ldots, (v-2w-4)/4\} \cup \{[w_1, (w+1)_1, (w+2)_1, w_2], [1_1, 1_2, (v-w-1)_1, (v-w)_1]\}$ we get a minimum covering of $K(v,w)$ where $P = M' \cup P_3$ with $E(M') = \{((2i-1)_1, (2i)_2), ((2i)_1, (2i-1)_2) \mid i = 1, 2, \ldots, (w-1)/2\} \cup \{(w_1, (w+1)_1), ((w+2)_1, w_2)\} \cup \{((w+4i-1)_1, (w+4i+2)_1), ((w+4i)_1, (w+4i+1)_1) \mid i = 1, 2, \ldots, (v-2w-4)/4\}$, and $E(P_3) = \{((v-w)_1, 1_1), (1_1, 2_1), (2_1, (v-w-1)_1)\}$, and so $|E(P)| = v/2 + 2$.

Case 17. Suppose $v \equiv 0 \pmod 4$ and $w \equiv 3 \pmod 8$, where $v - w \geq w$. As in Case 15, an optimal covering with padding P satisfies $|E(P)| \geq v/2$. Since $|(K(v,w))| + v/2 \equiv 1 \pmod 4$, then we need $|E(P)| \geq v/2 + 3$. Now $K(v,w) = K_{w-2} \cup K_{w-3,v-2w+2} \cup K_{w-3,2} \cup K_{v-2w+2,w-1} \cup (K_{w-2,w-2} \setminus M_1) \cup (K_{v-2w+2} \setminus M_2) \cup (v-2w+2)/2 \times C_4 \cup M_1 \cup M_2 \cup 2 \times K_2$ where the vertex set of K_{w-2} is $\{1_1, 2_1, \ldots, (w-2)_1\}$, the vertex set of $K_{w-3,v-2w+2}$ has partite sets $\{1_1, 2_1, \ldots, (w-3)_1\}$ and $\{(w-1)_1, w_1, \ldots, (v-w)_1\}$, the vertex set of $K_{w-3,2}$ has partite sets $\{1_1, 2_1, \ldots, (w-3)_1\}$ and $\{(w-1)_2, w_2\}$, the vertex set of $K_{v-2w+2,w-1}$ has partite sets $\{(w-1)_1, w_1, \ldots, (v-w)_1\}$ and $V_w \setminus \{(w-2)_2\}$, the vertex set of $K_{w-2,w-2}$ has partite sets $\{1_1, 2_1, \ldots, (w-2)_1\}$ and $\{1_2, 2_2, \ldots (w-2)_2\}$, the edge set of M_1 is $\{(i_1, i_2) \mid i = 1, 2, \ldots, w-2\}$, the vertex set of K_{v-2w+2} is $\{(w-1)_1, w_1, \ldots, (v-w)_1\}$, the edge set of M_2 is $\{((w+2i-3)_1, (w+2i-2)_1) \mid i = 1, 2, \ldots, (v-2w+2)/2\}$, $(v-2w+2)/2 \times C_4 = \{[(w-2)_1, (w+2i-3)_1, (w-2)_2, (w+2i-2)_1] \mid i = 1, 2, \ldots, (v-2w+2)/2\}$, and the edge set of $2 \times K_2$ is $\{((w-2)_1, (w-1)_2), ((w-2)_1, w_2)\}$. Since K_{w-2}, $K_{w-3,v-2w+2}$, $K_{w-3,2}$, $K_{v-2w+2,w-1}$, $(K_{w-2,w-2} \setminus M_1)$, and $(K_{v-2w+2} \setminus M_2)$ can be decomposed into copies of C_4, then by combining the blocks of the decompositions with 4-cycles $\{[(2i-1)_1, (2i-1)_2, (2i)_1, (2i)_2] \mid i = 1, 2, \ldots, (w-3)/2\} \cup \{[(w+4i-5)_1, (w+4i-2)_1, (w+4i-3)_1, (w+4i-4)_1] \mid i = 1, 2, \ldots, (v-2w+2)/4\} \cup \{[(w-2)_1, (w-1)_2, (w-1)_1, w_2], [(w-2)_1, (w-2)_2, w_1, (w-1)_1]\}$ we get a minimum covering of $K(v,w)$ where $P = M' \cup 3 \times K_2$, $E(M') = \{((2i-1)_1, (2i)_2), ((2i)_1, (2i-1)_2) \mid i = 1, 2, \ldots, (w-3)/2\} \cup \{((w+4i-5)_1, (w+4i-4)_1), ((w+4i-2)_1, (w+4i-3)_1) \mid i = 1, 2, \ldots, (v-2w+2)/4\} \cup \{((w-1)_1, (w-1)_2), ((w-1)_1, w_2)\}$, and $3 \times K_2 = \{(w_1, (w-2)_2), (w_1, (w-1)_1), ((w-1)_1, (w-2)_1)\}$ and so $|E(P)| = v/2 + 3$.

Case 18. Suppose $v \equiv 2 \pmod 4$ and $w \equiv 3 \pmod 8$, where $v - w \geq w$. As in Case 15, an optimal covering with padding P satisfies $|E(P)| \geq v/2$. Since $|E(K(v,w))| + v/2 \equiv 3 \pmod 4$, then we need $|E(P)| \geq v/2 + 1$. Now $K(v,w) = K_{w-2} \cup K_{w-3,v-2w+2} \cup K_{w-3,2} \cup K_{v-2w+2,w-1} \cup (K_{w-2,w-2} \setminus M_1) \cup (K_{v-2w+2} \setminus M_2) \cup (v-2w+2)/2 \times C_4 \cup M_1 \cup M_2 \cup 2 \times K_2$, as established in Case 17 (with the vertex sets as given in Case 17). Since K_{w-2}, $K_{w-3,v-2w+2}$, $K_{w-3,2}$, $K_{v-2w+2,w-1}$, $(K_{w-2,w-2} \setminus M_1)$, and $(K_{v-2w+2} \setminus M_2)$ can be decomposed into copies of C_4, then by combining the blocks of the decompositions with 4-cycles $\{[(2i-1)_1, (2i-1)_2, (2i)_1, (2i)_2] \mid i = 1, 2, \ldots, (w-3)/2\} \cup \{[(w+4i-3)_1, (w+4i)_1, (w+4i-1)_1, (w+4i-2)_1] \mid i = 1, 2, \ldots, (v-2w)/4\} \cup \{[(w-2)_1, (w-2)_2, w_1, (w-1)_1], [(w-2)_1, (w-1)_2, w_1, w_2]\}$ we get a minimum covering of $K(v,w)$ where $P = M' \cup 2 \times K_2$, $E(M') = \{((2i-1)_1, (2i)_2), ((2i)_1, (2i-1)_2) \mid i = 1, 2, \ldots, (w-3)/2\} \cup \{((w+4i-3)_1, (w+4i)_1), ((w+4i-1)_1, (w+4i-2)_1) \mid i = 1, 2, \ldots, (v-2w)/4\} \cup \{((w-2)_1, (w-1)_1), (w_1, (w-2)_2)\}$, and $E(2 \times K_2) = \{((w_1, (w-1)_2), (w_1, w_2)\}$, and so $|E(P)| = v/2 + 1$.

Case 19. Suppose $v \equiv 0 \pmod 4$ and $w \equiv 5 \pmod 8$, where $v - w \geq w$. As in Case 15, an optimal covering with padding P satisfies $|E(P)| \geq v/2$. Since $|E(K(v,w))| + v/2 \equiv 2 \pmod 4$, then we need $|E(P)| \geq v/2 + 2$. Now $K(v,w) = K_{w-4} \cup K_{w-5,v-2w+4} \cup (K_{w-4,w-4} \setminus M_1) \cup (K_{v-2w+4} \setminus M_2) \cup K_{v-2w+4,w-1} \cup K_{w-5,4} \cup \frac{v-2w+4}{2} \times C_4 \cup M_1 \cup M_2 \cup S_4$ where the vertex set of K_{w-4} is $\{1_1, 2_1, \ldots, (w-4)_1\}$, the vertex set of $K_{w-5,v-2w+4}$ has partite sets $\{1_1, 2_1, \ldots, (w-5)_1\}$ and $\{(w-3)_1, (w-2)_1, \ldots, (v-w)_1\}$, the vertex set of $K_{w-4,w-4}$ has partite sets $\{1_1, 2_1, \ldots, (w-4)_1\}$ and $\{1_2, 2_2, \ldots, (w-4)_2\}$, the edge set of M_1 is $\{(i_1, i_2) \mid i = 1, 2, \ldots, w-4\}$, the vertex set of K_{v-2w+4} is $\{(w-3)_1, (w-2)_1, \ldots, (v-w)_1\}$, the edge set of M_2 is $\{((w+2i-5)_1, (w+2i-4)_1) \mid i = 1, 2, \ldots, (v-2w+4)/2\}$, the vertex set of $K_{v-2w+4,w-1}$ has partite sets $\{(w-3)_1, (w-2)_1, \ldots, (v-w)_1\}$ and $\{1_2, 2_2, \ldots, (w-1)_2\}$, the vertex set of $K_{w-5,4}$ has partite sets $\{1_1, 2_1, \ldots, (w-5)_1\}$ and $\{(w-3)_2, (w-2)_2, (w-1)_2, w_2\}$, $(v-2w+4)/2 \times C_4 = \{[(w-4)_1, (w+2i-5)_1, (w-4)_2, (w+2i-4)_1] \mid i = 1, 2, \ldots, (v-2w+4)/2\}$, and the edge set of S_4 is $\{((w-4)_1, (w+i-4)_2) \mid i = 1, 2, 3, 4\}$. Since K_{w-4}, $K_{w-5,v-2w+4}$, $(K_{w-4,w-4} \setminus M_1)$, $(K_{v-2w+4} \setminus M_2)$, $K_{v-2w+4,w-1}$, and $K_{w-5,4}$ can be decomposed into copies of C_4, then by combining the blocks of the decompositions with 4-cycles $\{[(2i-1)_1, (2i-1)_2, (2i)_1, (2i)_2] \mid i = 1, 2, \ldots, (w-5)/2\} \cup \{[(w+4i-5)_1, (w+4i-2)_1, (w+4i-3)_1, (w+4i-4)_1] \mid i = 1, 2, \ldots, (v-2w+2)/4\} \cup \{[(w-4)_1, (w-4)_2, (w-2)_1, (w-3)_1], [(w-4)_1, (w-3)_2, w_1, (w-2)_2], \cup \{[(w-4)_1, (w-1)_2, w_1, w_2]\}$ we get a minimum covering of $K(v,w)$ where $P = M_1' \cup M_2' \cup 2 \times K_2 \cup S_4$, $E(M_1') = \{((2i-1)_1, (2i)_2), ((2i)_1, (2i-1)_2) \mid i = 1, 2, \ldots, (w-5)/2\}$, $E(M_2') = \{((w+4i-5)_1, (w+4i-2)_1), ((w+4i-3)_1, (w+4i-4)_1) \mid i = 1, 2, \ldots, (v-$

$2w+2)/4\}$, $E(2\times K_2)=\{((w-4)_1,(w-3)_1),((w-2)_1,(w-4)_2)\}$, and $E(S_4)=\{(w_1,(w+i-4)_1)\mid i=1,2,3,4\}$, and so $|E(P)|=v/2+2$.

Case 20. Suppose $v\equiv 2 \pmod 4$ and $w\equiv 5 \pmod 8$, where $v-w\geq w$. As in Case 15, an optimal covering with padding P satisfies $|E(P)|\geq v/2$. In this case, we assume the vertex set $K(v,w)$ is $V(K(v,w))=V'_{v-w}\cup V'_w$ where $V'_{v-w}=\{0_1,1_1,\ldots,(v-w-1)_1\}$ and $V'_w=\{0_2,1_2,\ldots,(w-1)_2\}$. Consider the following set of 4-cycles (where the vertex labels are reduced modulo w): $G=\{[j_1,(4i+j)_1,(1+j)_1,(4i-2+j)_1]\mid i=1,2,\ldots,(w-5)/8, j=1,2,\ldots,w\}\cup\{[(i-1)_1,(i-1)_2,i_1,((w-3+2i)/2)_1]\mid i=1,2,\ldots,w\}$. Then $K(v,w)=G\cup K_{w-1,v-2w}\cup(K_{v-2w}\setminus M_1)\cup K_{v-2w,w-1}\cup(K_{w,w}\setminus M_2)\cup(v-2w)/4\times C_4\cup(v-2w)/2\times C_4\cup M_1\cup M_2$ where the vertex set of $K_{w-1,v-2w}$ has partite sets $\{0_1,1_1,\ldots,(w-2)_1\}$ and $\{w_1,(w+1)_1,\ldots,(v-w-1)_1\}$, the vertex set of K_{v-2w} is $\{w_1,(w+1)_1,\ldots,(v-w-1)_1\}$, the edge set of M_1 is $\{((w+2i-2)_1,(w+2i-1)_1)\mid i=1,2,\ldots,(v-2w)/2\}$, the vertex set of $K_{v-2w,w-1}$ has partite sets $\{w_1,(w+1)_1,\ldots,(v-w-1)_1\}$ and $\{0_2,1_2,\ldots,(w-2)_2\}$, the vertex set of $K_{w,w}$ has partite sets $\{0_1,1_1,\ldots,(w-1)_1\}$ and $\{0_2,1_2,\ldots,(w-1)_2\}$, the edge set of M_2 is $\{(i_1,i_2)\mid i=0,1,\ldots,w-1\}$, $(v-2w)/4\times C_4=\{[(w+4i-4)_1,(w+4i-1)_1,(w+4i-2)_1,(w+4i-3)_1]\mid i=1,2,\ldots,(v-2w)/4\}$, and $(v-2w)/2\times C_4=\{[(w-1)_1,(w-2+2i)_1,(w-1)_2,(w-1+2i)_1]\mid i=1,2,\ldots,(v-2w)/2\}$. Since $K_{w-1,v-2w}$, $(K_{v-2w}\setminus M_1)$, $K_{v-2w,w-1}$, and $(K_{w,w}\setminus M_2)$ can be decomposed into copies of C_4, then there exists a minimum covering of $K(v,w)$ with padding $P=M'_1\cup M'_2$ where $E(M'_1)=\{((w+4i-4)_1,(w+4i-1)_1),((w+4i-3)_1,(w+4i-2)_1)\mid i=1,2,\ldots,(v-2w)/4\}$ and $E(M'_2)=\{(i_1,(i-1)_2)\mid i=1,2,\ldots,w-1\}\cup\{(0_1,(w-1)_2)\}$. So $|E(P)|=v/2$.

Case 21. Suppose $v\equiv 0 \pmod 4$ and $w\equiv 7 \pmod 8$, where $v-w\geq w$. As in Case 15, an optimal covering with padding P satisfies $|E(P)|\geq v/2$. Since $|E(K(v,w))|+v/2\equiv 3 \pmod 4$, then we need $|E(P)|\geq v/2+1$. In this case, we assume the vertex set of $K(v,w)$ is $V(K(v,w))=V'_{v-w}\cup V'_w$ where $V'_{v-w}=\{0_1,1_1,\ldots,(v-w-1)_1\}$ and $V'_w=\{0_2,1_2,\ldots,(w-1)_2\}$. Consider the set of 4-cycles (where vertex labels are reduced modulo $w-2$): $G=\{[j_1,(4i+j)_1,(1+j)_1,(4i-2+j)_1]\mid i=1,2,\ldots,(w-7)/8, j=1,2,\ldots,w-2\}\cup\{[(i-1)_1,(i-1)_2,i_1,((w-5+2i)/2)_1]\mid i=1,2,\ldots,w-2\}$. Then $K(v,w)=G\cup K_{w-3,v-2w+2}\cup(K_{v-2w+2}\setminus M_1)\cup K_{v-2w+2,w-1}K_{w-3,2}\cup(K_{w-2,w-2}\setminus M_2)\cup(v-2w+2)/2\times C_4\cup P_2\cup M_1\cup M_2$ where the vertex set of $K_{w-3,v-2w+2}$ has partite sets $\{0_1,1_1,\ldots,(w-4)_1\}$ and $\{(w-2)_1,(w-1)_1,\ldots,(v-w-1)_1\}$, the vertex set of K_{v-2w+2} is $\{(w-2)_1,(w-1)_1,\ldots,(v-w-1)_1\}$, the edge set of M_1 is $\{((w+2i-4)_1,(w+2i-3)_1)\mid i=1,2,\ldots,(v-2w+2)/2\}$, the vertex set of $K_{v-2w+2,w-1}$ has partite sets $\{(w-2)_1,(w-1)_1,\ldots,(v-w-1)_1\}$ and $\{0_2,1_2,\ldots,(w-4)_2,(w-2)_2,(w-1)_2\}$, the vertex set of $K_{w-3,2}$ has partite sets $\{0_1, 1_1, \ldots, (w-4)_1\}$ and $\{(w-2)_2, (w-1)_2\}$, the vertex set of $K_{w-2,w-2}$ has partite sets $\{0_1, 1_1, \ldots, (w-3)_1\}$ and $\{0_2, 1_2, \ldots, (w-3)_2\}$, the edge set of M_2 is $\{(i_1, i_2) \mid i = 0, 1, \ldots, w-3\}$, $(v-2w+2)/2 \times C_4 = \{[(w-3)_1, (w+2i-4)_1, (w-3)_2, (w+2i-3)_1] \mid i = 1, 2, \ldots, (v-2w+2)/2\}$, and the edge set of P_2 is $\{((w-3)_1, (w-2)_2), ((w-3)_1, (w-1)_2)\}$. Now $K_{w-3,v-2w+2}$, $(K_{v-2w+2} \backslash M_1)$, $K_{v-2w+2,w-1}$, $K_{w-3,2}$, and $(K_{w-2,w-2} \backslash M_2)$ can be decomposed into 4-cycles. Take the 4-cycles of such decompositions, along with G and $\{[(w+4i-6)_1, (w+4i-3)_1, (w+4i-4)_1, (w+4i-5)_1] \mid i = 1, 2, \ldots, (v-2w+2)/4\} \cup \{(w-3)_1, (w-2)_2, (w-2)_1, (w-1)_2]\}$. This is a minimum covering with $P = M_1' \cup M_2' \cup P_2'$ where $E(M_1') = \{(i_1, (i-1)_2) \mid i = 1, 2, \ldots, w-3\} \cup \{(0_1, (w-3)_2)\}$, $E(M_2') = \{((w+4i-6)_1, (w+4i-3)_1), ((w+4i-4)_1, (w+4i-5)_1) \mid i = 1, 2, \ldots, (v-2w+2)/4\}$, and $E(P_2') = \{((w-2)_1, (w-2)_2), ((w-2)_1, (w-1)_2\}$. So $|E(P)| = v/2 + 1$ and the covering is optimal.

Case 22. Suppose $v \equiv 2 \pmod 4$ and $w \equiv 7 \pmod 8$, where $v - w \geq w$. As in Case 15, an optimal covering with padding P satisfies $|E(P)| \geq v/2$. Since $|E((v,w))| + v/2 \equiv 1 \pmod 4$, then we need $|E(P)| \geq v/2 + 3$. Now $K(v,w) = K_{w-6} \cup K_{w-7,v-2w+6} \cup (K_{w-6,w-6} \backslash M_1) \cup (K_{v-2w+6} \backslash M_2) \cup K_{v-2w+6,w-1} \cup K_{w-7,6} \cup \frac{v-2w+6}{2} \times C_4 \cup S_6 \cup M_1 \cup M_2$ where the vertex set of K_{w-6} is $\{1_1, 2_1, \ldots, (w-6)_1\}$, the vertex set of $K_{w-7,v-2w+6}$ has partite sets $\{1_1, 2_1, \ldots, (w-7)_1\}$ and $\{(w-5)_1, (w-4)_1, \ldots, (v-w)_1\}$, the vertex set of $K_{w-6,w-6}$ has partite sets $\{1_1, 2_1, \ldots, (w-6)_1\}$ and $\{1_2, 2_2, \ldots, (w-6)_2\}$, the edge set of M_1 is $\{(i_1, i_2) \mid i = 1, 2, \ldots, w-6\}$, the vertex set of K_{v-2w+6} is $\{(w-5)_1, (w-4)_1, \ldots, (v-w)_1\}$, the edge set of M_2 is $\{((w+2i-7)_1, (w+2i-6)_1) \mid i = 1, 2, \ldots, (v-2w+6)/2\}$, the vertex set of $K_{v-2w+6,w-1}$ has partite sets $\{(w-5)_1, (w-4)_1, \ldots, (v-w)_1\}$ and $V_w \backslash \{(w-6)_2\}$, the vertex set of $K_{w-7,6}$ has partite sets $\{1_1, 2_1, \ldots, (w-7)_1\}$ and $\{(w-5)_2, (w-4)_2, \ldots, w_2\}$, $(v-2w+6)/2 \times C_4 = \{[(w-6)_1, (w+2i-7)_1, (w-6)_2, (w+2i-6)_1] \mid i = 1, 2, \ldots, (v-2w+6)/2\}$, and the edge set of S_6 is $\{((w-6)_1, (w+i-6)_2) \mid i = 1, 2, \ldots, 6\}$. Since K_{w-6}, $K_{w-7,v-2w+6}$, $(K_{w-6,w-6} \backslash M_1)$, $(K_{v-2w+6} \backslash M_2)$, $K_{v-2w+6,w-1}$, and $K_{w-7,6}$ can be decomposed into copies of C_4, then by combining the blocks of the decompositions with 4-cycles $\{[(2i-1)_1, (2i-1)_2, (2i)_1, (2i)_2] \mid i = 1, 2, \ldots, (w-7)/2\} \cup \{[(w+4i-7)_1, (w+4i-4)_1, (w+4i-5)_1, (w+4i-6)_1] \mid i = 1, 2, \ldots, (v-2w+4)/4\} \cup \{[(w-6)_1, (w-6)_2, (w-4)_1, (w-5)_1], [(w-6)_1, (w-5)_2, w_1, (w-4)_2]\} \cup \{[(w-6)_1, (w-3)_2, w_1, (w-2)_2], [(w-6)_1, (w-1)_2, w_1, w_2]\}$ we get a minimum covering of $K(v,w)$ where $P = M_1' \cup M_2' \cup 2 \times K_2 \cup S_6$ here $E(M_1') = \{((2i-1)_1, (2i)_2), ((2i)_1, (2i-1)_2) \mid i = 1, 2, \ldots, (w-7)/2\}$, $E(M_2') = \{((w+4i-7)_1, (w+4i-4)_1), ((w+4i-5)_1, (w+4i-6)_1) \mid i = 1, 2, \ldots, (v-2w+4)/4\}$, $E(2 \times K_2) = \{((w-6)_1, (w-5)_1), ((w-4)_1, (w-6)_2)\}$, and $E(S_6) = \{(w_1, (w+i-6)_2) \mid i = 1, 2, \ldots, 6\}$, and so $|E(P)| = v/2 + 3$. □

Acknowledgments

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References

[1] D. Bryant, D. Hoffman, C. Rodger, 5-cycle systems with holes, *Designs, Codes, and Cryptography*, **8** (1996), 103-108.

[2] D. Bryant, C. Rodger, E. Spicer, Embeddings of m-cycle systems and incomplete m-cycle systems: $m \leq 14$, *Discrete Mathematics*, **171** (1997), 55-75.

[3] M. Fort, G. Hedlund, Minimal coverings of pairs by triples, *Pacific Journal of Mathematics*, **8** (1958), 709-719.

[4] J. Kennedy, *Two Perfect Maximum Packings and Minimum Coverings of K_n with Hexagons*, Ph.D. Dissertation, Auburn University, USA (1995).

[5] E. Mendelsohn, A. Rosa, Embedding maximum packings of triples, *Congressus Numerantium*, **40** (1983), 235-247.

[6] J. Schönheim, A. Bialostocki, Packing and covering of the complete graph with 4-cycles, *Canadian Mathematics Bulletin*, **18**, No. 5 (1975), 703-708.