TRANSROTATIONAL DIRECTED TRIPLE SYSTEMS

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Abstract. A directed triple system of order v, denoted $DTS(v)$, is said to be k-transrotational if it admits an automorphism consisting of a fixed point, a transposition and k cycles of length $\frac{v-3}{k}$. In this paper, we give necessary and sufficient conditions for the existence of k-transrotational $DTS(v)$s for several values of k.

1. Introduction

A *directed triple system* of order v, denoted $DTS(v)$, is a v-element set, X, of points, together with a set, β, of ordered triples of elements of X, called *blocks*, such that any ordered pair of points of X occur in exactly one block of β. The notation $[x, y, z]$ will be used for the block containing the ordered pairs (x, y), (x, z), and (y, z). Hung and Mendelsohn [10] introduced directed triple systems as a generalization of Steiner triple systems and showed that a $DTS(v)$ exists if and only if $v \equiv 0$ or $1 \pmod 3$. An *automorphism* of a $DTS(v)$ is a permutation of X which fixes β. A permutation π of a v-element set is said to be of *type* $[\pi] = [p_1, p_2, \ldots, p_v]$ if the disjoint cyclic decomposition of π contains p_i cycles of length i. The *orbit* of a block under an automorphism, π, is the image of the block under the powers of π. A set of blocks, B, is said to be a *set of base blocks for a* $DTS(v)$ *under the permutation* π if the orbits of the blocks of B produce the $DTS(v)$ and exactly one block of B occurs in each orbit.

Several types of automorphisms have been explored in connection with the problem of determining the values v for which there are certain types of block designs of order v admitting the automorphism. In particular, a *cyclic* $DTS(v)$ admits an automorphism of type $[0, 0, \ldots, 1]$ and exists if and only if $v \equiv 1, 4$, or $7 \pmod{12}$ [4]. A $DTS(v)$ admitting an automorphism of type $[1, 0, \ldots, 0, k, \ldots, 0]$ is said to be *k-rotational.* A k-rotational $DTS(v)$ exists if and only if $kv \equiv 0 \pmod 3$, $v \equiv 1 \pmod k$ and $v \equiv 0$ or $1 \pmod 3$. A $DTS(v)$ admitting an automorphism of type $[3, 0, \ldots, 0, k, 0, \ldots, 0]$ is said to be *k-near-rotational.* A k-near-rotational $DTS(v)$ exists if and only if $k(v+2) \equiv 0 \pmod 3$, $v \equiv 3 \pmod k$, and $v \equiv 0$ or $1 \pmod 3$ [8].

Steiner triple systems, denoted STS, have been extensively explored in connection with this question. For a survey of results, see [5]. A cyclic $STS(v)$ exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$ [9, 11, 14]. The case of k-rotational STSs has been solved for $k = 1, 2, 3, 4$, and 6 [2, 12], and the case of k-near-rotational STSs has been solved for k divisible by 2 or 3 [6]. A block design admitting an automorphism of type $[1, 1, 0, \ldots, 0, k, 0, \ldots, 0]$ is said to be *k-transrotational.* A 1-transrotational $STS(v)$

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exists if and only if $v \equiv 1$, 7, 9, or 15 (mod 24) and a 2-transrotational $STS(v)$ exists if and only if $v \equiv 3$ or 19 (mod 24) [7]. A 3-transrotational $STS(v)$ exists if and only if $v \equiv 3$, 9, or 15 (mod 24) [1]. The purpose of this paper is to address the problem of existence for k-transrotational directed triple systems.

2. 1, 2, and 3-Transrotational Directed Triple Systems

We have the following necessary conditions:

Lemma 2.1 *If a k-transrotational $DTS(v)$ exists, then $k(v+2) \equiv 0$ (mod 3), $v \equiv 3$ (mod $2k$), and $v \equiv 0$ or 1 (mod 3).*

Proof. A k-transrotational $DTS(v)$ on the set $X = \{\infty, a, b\} \bigcup Z_N \times Z_k$ where $N = \frac{v-3}{k}$ admitting $\pi = (\infty)(a,b)(0_0, 1_0, \ldots, (N-1)_0) \cdots (0_{k-1}, 1_{k-1}, \ldots, (N-1)_{k-1})$ as an automorphism may contain blocks of the following forms only:

1. $[a, \infty, b]$ or $[b, \infty, a]$,
2. $[x_i, \infty, y_j]$ where $x_i, y_j \in Z_N \times Z_k$,
3. $[\infty, x_i, y_j]$ or $[x_i, y_j, \infty]$ where $i \neq j$ and $x_i, y_j \in Z_N \times Z_k$, and
4. $[x_i, a, y_j]$ or $[x_i, b, y_j]$ where $x_i, y_j, \in Z_N \times Z_k$.
5. $[a, x_i, y_j]$, $[b, x_i, y_j]$, $[x_i, y_j, a]$, or $[x_i, y_j, b]$ where $x_i, y_j \in Z_N \times Z_k$ and $i \neq j$ if $x \equiv y$ (mod 2), and
6. $[x_i, y_j, z_m]$ where $x_i, y_j, z_m \in Z_N \times Z_k$.

There must be some blocks of either type 4 or type 5. By applying the automorphism π^N to one of these blocks, we see that $\pi^N(a) = a$. Therefore, N is even and $v \equiv 3$ (mod $2k$). There are two blocks of the first type, which are images of each other under π. The orbits of blocks of the remaining types are of length N. The number of blocks in a $DTS(v)$ is $\frac{v(v-1)}{3}$ so a requirement for a k-transrotational $DTS(v)$ is $\frac{v(v-1)}{3} - 2 \equiv 0$ (mod N). That is, $k(v+2) \equiv 0$ (mod 3). The final condition on v follows trivially. ∎

In this section, we show that the necessary conditions of Lemma 2.1 are sufficient for $k = 1$, 2, and 3. The constructions are similar to those in another paper by the author [8].

Theorem 2.1 *A 1-transrotational $DTS(v)$ exists if and only if $v \equiv 1$ (mod 6).*

Proof. The necessary condition follows from Lemma 2.1.
<u>case 1.</u> If $v = 7$ then consider the blocks:

$[a, \infty, b]$, $[0_0, \infty, 2_0]$, $[0_0, a, 1_0]$, and $[0_0, b, 3_0]$.

<u>case 2.</u> If $v = 13$ then consider the blocks:

$[a,\infty,b]$, $[0_0,\infty,3_0]$, $[0_0,a,4_0]$, $[0_0,b,6_0]$, $[0_0,1_0,9_0]$, and $[0_0,2_0,7_0]$.

case 3. If $v \equiv 1 \pmod 6$, $v \geq 19$, say $v = 6t+1$ where $t \geq 3$, then consider the blocks:

$[a,\infty,b]$, $[0_0,\infty,(3t-1)_0]$, $[0_0,a,(2t-2)_0]$, $[0_0,b,(2t)_0]$,

$[0_0,(2r)_0,(3t-1+r)_0]$ for $r = 1,2,\ldots,t-2$, and

$[0_0,(2r-1)_0,(5t-3+r)_0]$ for $r = 1,2,\ldots,t$.

In each case, these are collections of base blocks for a 1-transrotational $DTS(v)$ under the automorphism π. ∎

We now turn our attention to the case $k = 2$.

Lemma 2.2 *If $v \equiv 7$ (mod 24), then there exists a 2-transrotational $DTS(v)$.*

Proof. A 3-rotational $DTS(7)$ is also 2-transrotational and such a system exists, as mentioned above. Now suppose $v \equiv 7 \pmod{24}$, say $v = 24t+7$, where $t > 0$. Consider the following collection of blocks, where the subscripts are reduced modulo 2:

$[a,\infty,b]$, $[0_i,\infty,(6t)_i]$ for $i \in Z_2$, $[0_i,a,(8t+2)_i]$ for $i \in Z_2$,

$[0_i,b,(10t+2)_i]$ for $i \in Z_2$, $[0_i,0_{i+1},(9t+2)_{i+1}]$ for $i \in Z_2$

$[0_i,(3t+2-r)_{i+1},(3t+1+r)_{i+1}]$ for $r = 1,2,\ldots,3k+1$ and for $i \in Z_2$,

$[0_i,(9t+2-r)_{i+1},(9t+2+r)_{i+1}]$ for $r = 1,2,\ldots,3k-1$ and for $i \in Z_2$,

$[0_i,(7t+1+r)_i,(7t+2-r)_i]$ for $r = 1,2,\ldots,k$ and for $i \in Z_2$, and

$[0_i,(9t+2-r)_i,(9t+2+r)_i]$ for $r = 1,2,\ldots,k-1$ and for $i \in Z_2$ (omit if $k = 1$).

These are the base blocks for a 2-transrotational $DTS(v)$ under π. ∎

Lemma 2.3 *If $v \equiv 19$ (mod 24), then there exists a 2-transrotational $DTS(v)$.*

Proof. If $v \equiv 19 \pmod{24}$ then there is a 2-transrotational $STS(v)$. Let B be a collection of base blocks for such a system with point set X and admitting the automorphism π. For each block (x,y,z) of B, take the blocks $[x,y,z]$ and $[z,y,x]$. This second collection of blocks is a collection of base blocks for a 2-transrotational $DTS(v)$. ∎

Lemmas 2.1 through 2.3 combine to give us:

Theorem 2.2 *A 2-transrotational $DTS(v)$ exists if and only if $v \equiv 7$ (mod 12).*

We handle 3-transrotational $DTS(v)$s in the following three lemmas. In each, the subscripts are reduced modulo 3. The first of these lemmas will make use of a particular structure. A (C,k)-*system* is a set of ordered pairs $\{(a_r,b_r)$ for $r=1,2,\ldots,k\}$ such that $b_r-a_r=r$ for $r=1,2,\ldots,k$ and $\bigcup_{r=1}^{k}\{a_r,b_r\}=\{1,2,\ldots,k,k+2,\ldots,2k+1\}$. A (C,k)-system exists if and only if $k\equiv 0$ or 3 (mod 4) [13].

Lemma 2.4 *If $v\equiv 3$ or 9 (mod 24), then there exists a 3-transrotational $DTS(v)$.*

Proof. Consider the blocks:

$[a,\infty,b]$, $[0_i,\infty,(\frac{v-3}{6})_i]$ for $i\in Z_3$, $[0_1,a,(\frac{v-3}{6})_0]$, $[0_2,a,(\frac{v-3}{6})_1]$, $[0_0,a,(\frac{v-3}{6})_2]$, $[0_0,b,(\frac{v-3}{6})_1]$,

$[0_1,b,(\frac{v-3}{6})_2]$, $[0_2,b,(\frac{v-3}{6})_0]$, $[0_0,0_1,0_2]$, $[0_2,0_1,0_0]$, and

$[0_i,r_i,(b_r)_{i+1}]$ and $[(b_r)_{i+1},r_i,0_i]$ for $r=1,2,\ldots,\frac{v-9}{6}$ and $i\in Z_3$ where $\{(a_r,b_r)$ for $r=1,2,\ldots,\frac{v-9}{6}\}$ is a $(C,\frac{v-9}{6})$-system.

These are the base blocks for a 3-transrotational $DTS(v)$ under π. ∎

Lemma 2.5 *If $v\equiv 15$ (mod 24), then there exists a 3-transrotational $DTS(v)$.*

Proof. Suppose $v\equiv 15$ (mod 24), say $v=24t+15$. Consider the blocks:

$[a,\infty,b]$, $[0_i,\infty,(4t+2)_i]$ for $i\in Z_3$, $[0_0,a,(2t+1)_1]$, $[0_1,a,(2t+1)_2]$, $[0_2,a,(2t+1)_0]$,

$[0_1,b,(6t+3)_0]$, $[0_2,b,(6t+3)_1]$, $[0_0,b,(6t+3)_2]$,

$[0_0,0_1,0_2]$, $[0_2,0_1,0_0]$,

$[0_i,(2r-1)_i,(6t+2+r)_{i+1}]$ and $[(6t+2+r)_{i+1},(2r-1)_i,0_i]$ for $r=1,2,\ldots,2t+1$ and for $i\in Z_3$,

$[0_i,(2r)_i,(2t+1+r)_{i+1}]$ and $[(2t+1+r)_{i+1},(2r)_i,0_i]$ for $r=1,2,\ldots,2t$ and for $i\in Z_3$.

These are base blocks for a 3-transrotational $DTS(v)$ under π. ∎

Lemma 2.6 *If $v\equiv 21$ (mod 24), then there exists a 3-transrotational $DTS(v)$.*

Proof. Suppose $v\equiv 21$ (mod 24), say $v=24t+21$. Consider the blocks:

$[a,\infty,b]$, $[0_i,\infty_1,(4t+3)_i]$ for $i\in Z_3$, $[0_0,a,(2t+2)_1]$, $[0_1,a,(2t+2)_2]$, $[0_2,a,(2t+2)_0]$,

$[0_1,b,(6t+4)_0]$, $[0_2,b,(6t+4)_1]$, $[0_0,b,(6t+4)_2]$, $[0_0,0_1,0_2]$, $[0_2,0_1,0_0]$,

$[0_i,(2r-1)_i,(6t+4+r)_{i+1}]$ and $[(6t+4+r)_{i+1},(2r-1)_i,0_i]$ for $r=1,2,\ldots,2t+1$ and for $i\in Z_3$, and

$[0_i,(2r)_i,(2t+2+r)_{i+1}]$ and $[(2t+2+r)_{i+1},(2r)_i,0_i]$ for $r=1,2,\ldots,2t+1$ and for $i\in Z_3$.

These are the base blocks for a 3-transrotational $DTS(v)$ under π. ∎

Lemmas 2.1 and 2.4 through 2.6 combine to give us:

Theorem 2.3 *A 3-transrotational $DTS(v)$ exists if and only if $v \equiv 3 \pmod 6$.*

3. Additional Transrotational Directed Triple Systems

In this section, we use the results of the previous section to generate k-transrotational directed triple systems for several infinite classes of values of k. If π is of type $[1,1,0,\ldots,0,k,0,$
$\ldots,0]$ then π^N is of type $[1,1,0,\ldots,0,kn,0,\ldots,0]$ provided that $n \mid \left(\frac{v-3}{k}\right)$ and n is odd. We take advantage of this in the following corollaries. In each, the necessary conditions follow from Lemma 2.1.

Corollary 3.1 If $k \equiv 1 \pmod 6$ then a k-transrotational $DTS(v)$ exists if and only if $v \equiv 3 + 4k \pmod{6k}$.

Proof. If $v \equiv 3 + 4k \pmod{6k}$, then there exists a 1-transrotational $DTS(v)$ with the relevant automorphism π. By considering π^k, we see that this $DTS(v)$ is also k-transrotational. ∎

Corollary 3.2 If $k \equiv 3 \pmod 6$ then a k-transrotational $DTS(v)$ exists if and only if $v \equiv 3 \pmod{2k}$.

Proof. If $v \equiv 3 \pmod{2k}$, then there exists a 3-transrotational $DTS(v)$ with the relevant automorphism π. By considering $\pi^{k/3}$, we see that this $DTS(v)$ is also k-transrotational. ∎

Corollary 3.3 If $k \equiv 5 \pmod 6$ then a k-transrotational $DTS(v)$ exists if and only if $v \equiv 3 + 2k \pmod{6k}$.

Proof. As with Corollary 3.1, sufficiency follows from 1-transrotational $DTS(v)$s. ∎

Corollary 3.4 If $k \equiv 2 \pmod{12}$ then a k-transrotational $DTS(v)$ exists if and only if $v \equiv 3 + 2k \pmod{6k}$.

Proof. If $v \equiv 3 + 2k \pmod{6k}$, then there exists a 2-transrotational $DTS(v)$ with the relevant automorphism π. By considerding $\pi^{k/2}$, we see that this $DTS(v)$ is also k-transrotational. ∎

Corollary 3.5 If $k \equiv 10 \pmod{12}$ then a k-transrotational $DTS(v)$ exists if and only if $v \equiv 3 + 4k \pmod{6k}$.

Proof. As with Corollary 3.4, sufficiency follows from 2-transrotational $DTS(v)$s. ∎

Corollaries 3.1 through 3.5 show that the necessary conditions of Lemma 2.1 are sufficient for all k except $k \equiv 0 \pmod 4$ and $k \equiv 6 \pmod{12}$. We leave these cases open.

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