Restricted and Unrestricted Hexagon Coverings of the Complete Bipartite Graph

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"...the Nobility, of whom there are several degrees, beginning at Six-Sided Figures, or Hexagons..." — E. A. Abbott, *Flatland*

Abstract. A *minimal covering* of a simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$, $V(g_i) \subset V(G)$, $G \subset \cup_{i=1}^n g_i$, and $|\cup_{i=1}^n E(g_i) \setminus E(G)|$ is minimal (the graph $\cup_{i=1}^n g_i$ may not be simple and $\cup_{i=1}^n E(g_i)$ may be a multiset). Some studies have been made of covering the complete graph, in which case an added condition of "$E(g_i) \subset E(G)$ for all i" implies no additional restrictions. However, if G is not the complete graph then this condition may have implications. We will give necessary and sufficient conditions for minimal coverings (as defined above, without the added restriction) of $K_{m,n}$ with 6-cycles, which we call minimal *unrestricted coverings.* We also give necessary and sufficient conditions for minimal coverings of $K_{m,n}$ with 6-cycles with the added condition $E(g_i) \subset E(G)$ for all i, and call these minimal *restricted coverings.*

1. Introduction

A *decomposition* of a simple graph G into isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, and $\cup_{i=1}^n E(g_i) = E(G)$, where $V(G)$ is the vertex set of graph G and $E(G)$ is the edge set of graph G. We will refer to such a decomposition as a "g decomposition of G." In the event that a g decomposition of G does not exist, we can ask the question "How close can we get to a g decomposition of G?" There are two approaches to this question: packings and coverings.

A *maximal packing* of a simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$ and $V(g_i) \subset V(G)$ for all i, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, $\cup_{i=1}^n g_i \subset G$, and $|E(G) \setminus \cup_{i=1}^n E(g_i)|$ is minimal. The set of edges for the *leave*, L, of the packing is $E(L) = E(G) \setminus \cup_{i=1}^n E(g_i)$.

A covering of a graph G with graph g is a collection of copies of g such that each edge of G appears in at least one copy of g. Coverings, though studied less than packings, have been studied for $G = K_v$ and

$g = C_3$ [3], $g = C_4$ [8], and $g = C_6$ (a "hexagon") [6]. If $G = K_v$ then obviously the edge sets of the copies of g are subsets of the edge set of G. However, if G is not the complete graph, we are faced with the question: Do we allow the copies of g to contain edges that are not in G, or do we view such edges as "forbidden" and hence require the copies of g to avoid such edges? Therefore we define two kinds of coverings. A *minimal unrestricted covering* of simple graph G with isomorphic copies of a graph g is a set $\{g_1, g_2, \ldots, g_n\}$ where $g_i \cong g$, $V(g_i) \subset V(G)$, $G \subset \cup_{i=1}^n g_i$, and $|\cup_{i=1}^n E(g_i) \setminus G|$ is minimal (the graph $\cup_{i=1}^n g_i$ may not be simple and $\cup_{i=1}^n E(g_i)$ may be a multiset). A *minimal restricted covering* of simple graph G with isomorphic copies of a graph g is a minimal unrestricted covering that also satisfies $E(g_i) \subset E(G)$ for all i. The purpose of this paper is to give necessary and sufficient conditions for both restricted and unrestricted coverings of the complete bipartite graph, $G = K_{m,n}$, with isomorphic copies of the hexagon, $g = C_6$.

2. Some Previous Results Concerning Hexagons

The following is well known:

Theorem 2.1 *The complete graph K_v can be decomposed into hexagons if and only if $v \equiv 1$ or $9 \pmod{12}$.*

Conditions for a hexagon decomposition of $K_{m,n}$ were given by Sotteau [9]:

Theorem 2.2 *The complete bipartite graph $K_{m,n}$ can be decomposed into hexagons if and only if $m \equiv 0 \pmod 6$ and $n \equiv 0 \pmod 2$, $n \geq 4$.*

Conditions for a hexagon decomposition of $K_{n,n}$ minus a matching we given in [2]:

Theorem 2.3 *A hexagon decomposition of $K_{n,n} \setminus M$, where M is a perfect matching of $K_{n,n}$, exists if and only if $n \equiv 1$ or $3 \pmod 6$.*

Maximal hexagon packings of $K_{m,n}$ were given in [2]:

Theorem 2.4 *A maximal hexagon packing of $K_{m,n}$ with leave L satisfies (1) when $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$, $|E(L)| = m + k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| - (m + k) \equiv 0 \pmod 6$, (2) when $m \equiv n \equiv 1 \pmod 2$ and $m \geq n$, $|E(L)| = m + k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| - (m + k) \equiv 0 \pmod 6$, (3) when $m \equiv 0 \pmod 6$ and $n \equiv 0 \pmod 2$, $|E(L)| = 0$,*

(4) *when* $m \equiv n \equiv 2$ *(mod 6) or* $m \equiv n \equiv 4$ *(mod 6), then* $|E(L)| = 4$*, and*
(5) *when* $m \equiv 2$ *(mod 6) and* $n \equiv 4$ *(mod 6), then* $|E(L)| = 8$.

3. Restricted Hexagon Coverings of $K_{m,n}$

Throughout this paper, unless stated otherwise, we denote the partite sets of $K_{m,n}$ as V_m and V_n where $V_m = \{1_1, 2_1, \ldots, m_1\}$ and $V_n = \{1_2, 2_2, \ldots, n_2\}$. We denote the hexagon (or "6-cycle") with edge set $\{(a,b), (b,c), (c,d), (d,e), (e,f), (f,a)\}$ by any cyclic shift of $[a,b,c,d,e,f]$. Since the hexagon is not a subgraph of $K_{m,n}$ when either m or n is less than 3, we assume in this section that both m and n are greater than or equal to 3.

Lemma 3.1 *A minimal restricted hexagon covering of* $K_{m,n}$ *where* m *and* n *are even,* $m \geq 4$ *and* $n \geq 4$*, has a padding* P *satisfying:*
(1) $|E(P)| = 0$ *when* $m \equiv 0$ *(mod 6),*
(2) $|E(P)| = 2$ *when* $m \equiv n \equiv 2$ *(mod 6) or* $m \equiv n \equiv 4$ *(mod 6), and*
(3) $|E(P)| = 4$ *when* $m \equiv 2$ *(mod 6) and* $n \equiv 4$ *(mod 6).*

Proof. We consider cases.
Case 1. Suppose $m \equiv 0$ (mod 6), $n \equiv 0$ (mod 2), and $n \geq 4$. Then $K_{m,n}$ can be decomposed into hexagons by Theorem 2.2 and in a minimal covering $|E(P)| = 0$.
Case 2. Suppose $m \equiv n \equiv 2$ (mod 6), $m, n \geq 4$. Now $|E(K_{m,n})| \equiv 4$ (mod 6), so it is necessary that a covering have a padding with $|E(P)| \geq 2$. Now $K_{m,n} = K_{m-8,n} \cup K_{8,n-8} \cup K_{8,8}$ where the partite sets of $K_{m-8,n}$ are $\{9_1, 10_1, \ldots, m_1\}$ and V_n, the partite sets of $K_{8,n-8}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{9_2, 10_2, \ldots, n_2\}$, and the partite sets of $K_{8,8}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 8_2\}$. Now $K_{m-8,n}$ and $K_{8,n-8}$ can be decomposed into hexagons by Theorem 2.2. Next we note that there is a restricted hexagon covering of $K_{8,8}$, namely the set $\{[2_1, 2_2, 4_1, 8_2, 1_1, 3_2]$, $[3_1, 3_2, 5_1, 1_2, 2_1, 4_2]$, $[5_1, 5_2, 7_1, 3_2, 4_1, 6_2]$, $[7_1, 7_2, 1_1, 5_2, 6_1, 8_2]$, $[8_1, 3_2, 6_1, 1_2, 4_1, 7_2]$, $[6_1, 6_2, 1_1, 4_2, 5_1, 7_2]$, $[1_1, 1_2, 3_1, 8_2, 5_1, 3_2]$, $[2_1, 5_2, 8_1, 6_2, 3_1, 7_2]$, $[8_1, 2_2, 7_1, 6_2, 2_1, 8_2]$, $[8_1, 2_2, 6_1, 4_2, 7_1, 1_2]$, $[8_1, 2_2, 3_1, 5_2, 4_1, 4_2]\}$. This is a minimal restricted covering of $K_{m,n}$ with padding P where $E(P) = \{(8_1, 2_2), (8_1, 2_2)\}$ and so $|E(P)| = 2$.
Case 3. Suppose $m \equiv n \equiv 4$ (mod 6). As in Case 2, a packing with padding P satisfies $|E(P)| \geq 2$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-4} \cup K_{4,4}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-4}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2, 4_2\}$, and the partite sets of $K_{4,4}$ are $\{1_1, 2_2, 3_1, 4_1\}$ and $\{1_2, 2_2, 3_2, 4_2\}$. Now $K_{m-4,n}$ and $K_{4,n-4}$ can be decomposed into hexagons by Theorem 2.2. Next we note that there is a restricted hexagon covering of $K_{4,4}$, namely the

set $\{[1_1, 1_2, 3_1, 4_2, 4_1, 2_2], [1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [1_1, 1_2, 4_1, 3_2, 2_1, 4_2]\}$. This is a minimal restricted covering of $K_{m,n}$ with padding P where $E(P) = \{2 \times (1_1, 1_2)\}$ and so $|E(P)| = 2$.
Case 4. Suppose $m \equiv 2 \pmod 6$, $m \geq 8$, and $n \equiv 4 \pmod 6$. Now $|E(K_{m,n}| \equiv 2 \pmod 6$, so it is necessary that a covering have a padding with $|E(P)| \geq 4$. Now $K_{m,n} = K_{8,n-4} \cup K_{m-8,n} \cup K_{8,4}$ where the partite sets of $K_{8,n-4}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2, 4_2\}$, the partite sets of $K_{m-8,n}$ are $\{9_1, 10_1, \ldots, m_1\}$ and V_n, and the partite sets of $K_{8,4}$ are $\{1_1, 2_2, \ldots, 8_1\}$ and $\{1_2, 2_2, 3_2, 4_2\}$. Now $K_{8,n-4}$ and $K_{m-8,n}$ can be decomposed into hexagons by Theorem 2.2. Next we note that there is a restricted hexagon covering of $K_{8,4}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [4_1, 1_2, 3_1, 4_2, 2_1, 3_2], [5_1, 1_2, 6_1, 2_2, 7_1, 3_2], [8_1, 1_2, 7_1, 4_2, 6_1, 3_2], [1_1, 4_2, 8_1, 1_2, 5_1, 2_2], [4_1, 4_2, 5_1, 1_2, 8_1, 2_2]\}$. This is a minimal restricted covering of $K_{m,n}$ with padding P where $E(P) = \{(5_1, 1_2), (5_1, 1_2), (8_1, 1_2), (8_1, 1_2)\}$ and so $|E(P)| = 4$. ■

Lemma 3.2 *A minimal restricted hexagon covering of $K_{m,n}$ where m is even and n is odd ($m \geq 4, n \geq 3$) has a padding P satisfying $|E(P)| = m+k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m+k) \equiv 0 \pmod 6$.*

Proof. Since each vertex of V_m is of odd degree in $K_{m,n}$, in the padding of a covering each of these vertices will be of odd degree. Therefore in a restricted covering of $K_{m,n}$ with padding P, it is necessary that $|E(P)| \geq m$. Since a covering yields a decomposition of $K_{m,n} \cup P$, then it is necessary that $|E(K_{m,n})| + |E(P)| \equiv 0 \pmod 6$.
Case 1. First, suppose $m \equiv 0 \pmod 6$ and $n = 5$. Consider the set of hexagons $\{[(1+6i)_1, 1_2, (2+6i)_1, 2_2, (3+6i)_1, 3_2], [(4+6i)_1, 3_2, (5+6i)_1, 4_2, (6+6i)_1, 5_2], [(3+6i)_1, 1_2, (5+6i)_1, 2_2, (4+6i)_1, 4_2], [(1+6i)_1, 2_2, (6+6i)_1, 3_2, (2+6i)_1, 4_2], [(3+6i)_1, 1_2, (6+6i)_1, 3_2, (5+6i)_1, 5_2], [(1+6i)_1, 1_2, (4+6i)_1, 2_2, (2+6i)_1, 5_2] \mid i = 0, 1, \ldots, m/6-1\}$. This is a restricted hexagon covering of $K_{m,n}$ with padding P satisfying $E(P) = \{((1+6i)_1, 1_2), ((2+6i)_1, 2_2), ((3+6i)_1, 1_2), ((4+6i)_1, 2_2), ((5+6i)_1, 3_2), ((6+6i)_1, 3_2) \mid i = 0, 1, \ldots, m/6-1\}$, and so $|E(P)| = m$ and the restricted covering is minimal.

Next, suppose $m \equiv 0 \pmod 6$, $n \equiv 1 \pmod 2$, and $n \neq 5$. Now $K_{m,n} = K_{m,n-3} \cup \frac{m}{6} \times K_{6,3}$ where the partite sets of $K_{m,n-3}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2\}$, and $E(\frac{m}{6} \times K_{6,3}) = \{[(1+6i)_1, 1_2, (4+6i)_1, 2_2, (2+6i)_1, 3_2], [(3+6i)_1, 2_2, (5+6i)_1, 1_2, (6+6i)_1, 3_2], [(3+6i)_1, 1_2, (5+6i)_1, 3_2, (4+6i)_1, 2_2], [(1+6i)_1, 2_2, (6+6i)_1, 1_2, (2+6i)_1, 3_2] \mid i = 0, 1, \ldots, \frac{m}{6}-1\}$. Now $K_{m,n-3}$ can be decomposed into hexagons by Theorem 2.2. Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding P where $E(P) = \{((1+6i)_1, 3_2), ((2+6i)_1, 3_2), ((3+6i)_1, 2_2), ((4+6i)_1, 2_2), ((5+6i)_1, 1_2), ((6+6i)_1, 1_2) \mid i = 0, 1, \ldots, \frac{m}{6}-1\}$ and so $|E(P)| = m$ and the restricted covering is minimal.

Case 2. Suppose $m \equiv 2 \pmod 6$, $m \geq 8$, $n \equiv 1 \pmod 6$, and $n \geq 7$. Now $K_{m,n} = K_{m-8,n} \cup K_{8,n-7} \cup K_{8,7}$ where the partite sets of $K_{m-8,n}$ are $V_m \setminus \{1_1, 2_1, \ldots, 8_1\}$ and V_n, the partite sets of $K_{8,n-7}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $V_n \setminus \{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of $K_{8,7}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$. Now $K_{m-8,n}$ has a restricted hexagon covering with padding P where $|E(P)| = m - 8$ (by Case 1) and $K_{8,n-7}$ can be decomposed into hexagons by Theorem 2.2. Next, we note that there is a restricted hexagon covering of $K_{8,7}$, namely the set $\{[2_1, 4_2, 3_1, 5_2, 4_1, 6_2]$, $[6_1, 4_2, 7_1, 5_2, 8_1, 6_2]$, $[1_1, 1_2, 2_1, 2_2, 3_1, 3_2]$, $[5_1, 1_2, 6_1, 2_2, 7_1, 3_2]$, $[1_1, 2_2, 3_1, 1_2, 4_1, 7_2]$, $[5_1, 2_2, 7_1, 1_2, 8_1, 7_2]$, $[1_1, 4_2, 4_1, 3_2, 2_1, 5_2]$, $[5_1, 4_2, 8_1, 3_2, 6_1, 5_2]$, $[1_1, 6_2, 3_1, 7_2, 2_1, 1_2]$, $[5_1, 6_2, 7_1, 7_2, 6_1, 1_2]$, $[4_1, 2_2, 8_1, 3_2, 3_1, 1_2]\}$ with padding P_2 satisfying $E(P_2) = \{(3_1, 2_2), (7_1, 2_2), (1_1, 1_2), (2_1, 1_2), (5_1, 1_2), (6_1, 1_2), (8_1, 3_2), (3_1, 3_2), (3_1, 1_2), (4_1, 1_2)\}$ and so $|E(P_2)| = 10$. Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m + 2$ and the restricted covering is minimal.

Case 3. Suppose $m \equiv 2 \pmod 6$, $n \equiv 3 \pmod 6$, and $m \geq 8$. Now $K_{m,n} = K_{m-8,n} \cup K_{8,n-3} \cup K_{8,3}$ where the partite sets of $K_{m-8,n}$ are $V_m \setminus \{1_1, 2_1, \ldots, 8_1\}$ and V_n, the partite sets of $K_{8,n-3}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2\}$, and the partite sets of $K_{8,3}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, 3_2\}$. Now $K_{m-8,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = m-8$ (by Case 1), and there is a hexagon decomposition of $K_{8,n-3}$ by Theorem 2.2.

Next, we note that there is a restricted hexagon covering of $K_{8,3}$, namely the set $\{[1_1, 3_2, 3_1, 2_2, 2_1, 1_2]$, $[4_1, 1_2, 5_1, 2_2, 6_1, 3_2]$, $[6_1, 1_2, 7_1, 3_2, 8_1, 2_2]$, $[2_1, 1_2, 7_1, 2_2, 5_1, 3_2]$, $[1_1, 1_2, 8_1, 3_2, 4_1, 2_2]$, $[3_1, 1_2, 4_1, 2_2, 5_1, 3_2]\}$ with padding P_2 satisfying $E(P_2) = \{(6_1, 2_2), (2_1, 1_2), (7_1, 1_2), (5_1, 2_2), (1_1, 1_2), (4_1, 3_2), (8_1, 3_2), (3_1, 3_2), (5_1, 3_2), (5_1, 2_2), (4_1, 2_2), (4_1, 1_2)\}$ and so $|E(P_2)| = 12$. Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m + 4$ and the restricted covering is minimal.

Case 4. Suppose $m \equiv 2 \pmod 6$, $n \equiv 5 \pmod 6$, and $m \geq 8$. Now $K_{m,n} = K_{m-8,n} \cup K_{8,n-5} \cup K_{8,5}$ where the partite sets of $K_{m-8,n}$ are $V_m \setminus \{1_1, 2_1, \ldots, 8_1\}$ and V_n, the partite sets of $K_{8,n-5}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $V_n \setminus \{1_2, 2_2, \ldots, 5_2\}$, and the partite sets of $K_{8,5}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 5_2\}$. Now $K_{m-8,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = m - 8$ (by Case 1), and there is a hexagon decomposition of $K_{8,n-5}$ by Theorem 2.2. Next, we note that there is a restricted hexagon covering of $K_{8,5}$, namely the set $\{[1_1, 1_2, 2_1, 3_2, 3_1, 2_2]$, $[1_1, 3_2, 4_1, 1_2, 2_1, 4_2]$, $[5_1, 1_2, 6_1, 3_2, 7_1, 2_2]$, $[1_1, 1_2, 3_1, 4_2, 4_1, 5_2,]$ $[5_1, 1_2, 7_1, 4_2, 8_1, 5_2]$, $[2_1, 2_2, 4_1, 3_2, 3_1, 5_2]$, $[6_1, 2_2, 8_1, 3_2, 7_1, 5_2]$, $[5_1, 3_2, 8_1, 1_2, 6_1, 4_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 1_2), (3_1, 3_2), (5_1, 1_2), (7_1, 3_2), (6_1, 1_2), (8_1, 3_2), (2_1, 1_2), (4_1, 3_2)\}$ and so $|E(P_2)| = 8$. Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where

$|E(P)| = m$.

Case 5. Suppose $m \equiv 4 \pmod 6$, $n \equiv 1 \pmod 6$, and $n \geq 7$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-7} \cup K_{4,7}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-7}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of $K_{4,7}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, \ldots 7_2\}$. Now $K_{m-4,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = m - 4$ (by Case 1), and there is a hexagon decomposition of $K_{4,n-7}$ by Theorem 2.2. Next, we note that there is a restricted hexagon covering of $K_{4,7}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2]$, $[2_1, 5_2, 3_1, 6_2, 4_1, 7_2]$, $[1_1, 3_2, 2_1, 4_2, 4_1, 5_2]$, $[1_1, 6_2, 2_1, 4_2, 3_1, 7_2]$, $[1_1, 2_2, 4_1, 3_2, 2_1, 4_2]$, $[2_1, 5_2, 3_1, 1_2, 4_1, 6_2]\}$, with padding P_2 satisfying $E(P_2) = \{(1_1, 3_2)$, $(2_1, 4_2)$, $(2_1, 3_2)$, $(2_1, 4_2)$, $(2_1, 5_2)$, $(2_1, 6_2)$, $(3_1, 5_2)$, $(4_1, 6_2)\}$ and so $|E(P_2)| = 8$. Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m + 4$.

Case 6. Suppose $m \equiv 4 \pmod 6$, $n \equiv 3 \pmod 6$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-3} \cup K_{4,3}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-3}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2\}$, and the partite sets of $K_{4,3}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, 3_2\}$. Now $K_{m-4,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = m - 4$ (by Case 1), and there is a hexagon decomposition of $K_{4,n-3}$ by Theorem 2.2. Next, we note that there is a restricted hexagon covering of $K_{4,3}$, namely the set $\{[1_1, 1_1, 2_1, 2_2, 3_1, 3_2]$, $[1_1, 1_2, 2_1, 3_2, 4_1, 2_2]$, $[2_1, 2_2, 3_1, 1_2, 4_1, 3_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 1_2)$, $(2_1, 1_2)$, $(2_1, 2_2)$, $(2_1, 3_2)$, $(3_1, 2_2)$, $(4_1, 3_2)\}$ and so $|E(P_2)| = 6$. Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m + 2$.

Case 7. Suppose $m \equiv 4 \pmod 6$, $n \equiv 5 \pmod 6$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-5} \cup K_{4,5}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-5}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, \ldots, 5_2\}$, and the partite sets of $K_{4,5}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, \ldots 5_2\}$. Now $K_{m-4,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = m - 4$ (by Case 1), and there is a hexagon decomposition of $K_{4,n-5}$ by Theorem 2.2. Next, we note that there is a restricted hexagon covering of $K_{4,5}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2]$, $[1_1, 2_2, 4_1, 5_2, 3_1, 4_2]$, $[2_1, 3_2, 3_1, 1_2, 4_1, 4_2]$, $[1_1, 1_2, 4_1, 3_2, 2_1, 5_2]\}$ with padding P_2 satisfying $E(P_2) = \{(3_1, 3_2)$, $(1_1, 1_2)$, $(2_1, 3_2)$, $(4_1, 1_2),\}$ and so $|E(P_2)| = 4$. Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$. ■

Lemma 3.3 *A minimal restricted hexagon covering of $K_{m,n}$ where $m = n \equiv 5 \pmod 6$ has padding P satisfying $|E(P)| = m$.*

Proof. Each vertex of V_m is of odd degree, so in a minimal covering, as in Lemma 3.2, it is necessary that $|E(P)| \geq m$. In the constructions for this case, we assume that $V_m = \{0_1, 1_1, \ldots, (m-1)_1\}$ and $V_n = \{0_2, 1_2, \ldots, (n-1)_2\}$. In each of the following two cases, we reduce the vertex labels modulo m.

Case 1. Suppose $m = n \equiv 5 \pmod{12}$. Consider the set of hexagons $\{[j_1, (4+j)_2, (1+j)_1, (1+j)_2, (m-1+j)_1, j_2]\} \cup \{[j_1, (11+12i+j)_2, (2+j)_1, (9+12i+j)_2, (1+j)_1, (6+12i+j)_2], [j_1, (16+12i+j)_2, (1+j)_1, (15+12i+j)_2, (2+j)_1, (12+12i+j)_2] \mid i = 0, 1, \ldots, (n-17)/12; j = 0, 1, 2, \ldots, m-1\}$ This is a minimal hexagon covering of $K_{m,n}$ with padding P where $E(P) = \{(i_1, i_2) \mid i = 0, 1, \ldots, m-1\}$, and so $|E(P)| = m$.

Case 2. Suppose $m = n \equiv 11 \pmod{12}$. Consider the set of hexagons $\{[j_1, (6+12i+j)_2, (2+j)_1, (4+12i+j)_2, (1+j)_1, (1+12i+j)_2], [j_1, (11+12i+j)_2, (1+j)_1, (10+12i+j)_2, (2+j)_1, (7+12i+j)_2] \mid i = 0, 1, \ldots, (n-11)/12; j = 0, 1, 2, \ldots, m-1\}$ This is a minimal hexagon covering of $K_{m,n}$ with padding P where $E(P) = \{(i_1, i_2) \mid i = 0, 1, \ldots, m-1\}$, and so $|E(P)| = m$. ■

Lemma 3.4 *A minimal restricted hexagon covering of $K_{m,n}$ where m and n are both odd, $m \geq n \geq 3$, has a padding P satisfying $|E(P)| = m + k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m+k) \equiv 0$ (mod 6).*

Proof. The necessary conditions follow as in Lemma 3.2. We now establish sufficiency.

Case 1. Suppose $m \equiv n \equiv 1 \pmod 6$, $m \geq n \geq 7$. Now $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1 \ldots, m_1\}$ and V_n, and the partite sets of $K_{n,n}$ are $\{1_1, 2_1, \ldots, n_1\}$ and V_n. By Theorem 2.3, there is a decomposition of $K_{n,n} \setminus M$ where M is a perfect matching of $K_{n,n}$, say $E(M) = \{(i_1, 1_2) \mid i = 1, 2, \ldots, n\}$. Taking the collection of hexagons for such a decomposition along with the set of hexagons $\{[(1+3i)_1, (3+3i)_2, (3+3i)_1, (2+3i)_2, (2+3i)_1, (1+3i)_2] \mid i = 0, 1, \ldots, \frac{n-4}{3}\}$, and we see that $K_{n,n} \setminus \{(n_1, n_2)\}$ has a hexagon covering with padding P_1 where $E(P_1) = \{((1+3i)_1, (3+3i)_2), ((2+3i)_1, (1+3i)_2), (3+3i)_1, (2+3i)_2) \mid i = 0, 1, \ldots, \frac{n-4}{3}\}$ and so $|E(P_1)| = n-1$. We cover edge (n_1, n_2) with hexagon $[n_1, n_2, (n-1)_1, (n-1)_2, (n-2)_1, (n-2)_2]$ and add to the padding the edges in $E(P_2) = \{(n_2, (n-1)_1), ((n-1)_1, (n-1)_2), ((n-1)_2, (n-2)_1), ((n-2)_1, (n-2)_2), (n_1, (n-2)_2)\}$ and so $|E(P_2)| = 5$. By Lemma 3.2 Case 1, there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_3 where $|E(P_3)| = m-n$. Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2 \cup P_3$ where $|E(P)| = m+4$.

Case 2. Suppose $m \equiv 1 \pmod 6$, $n \equiv 3 \pmod 6$, $m \geq n$. Now $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{n,n}$ are $\{(n+1)_1, (n+2)_1 \ldots, m_1\}$ and V_n, and the partite sets of $K_{m-n,n}$ are $\{1_1, 2_1, \ldots, n_1\}$ and V_n. By Theorem 2.3, there is a decomposition of $K_{n,n} \setminus M$ where M is a perfect matching of $K_{n,n}$, say $E(M) = \{(i_1, i_2) \mid i = 1, 2, \ldots, n\}$. Taking the collection of hexagons for such a decomposition along with the set of hexagons $\{[(1+3i)_1, (3+3i)_2, (3+3i)_1, (2+3i)_2, (2+3i)_1, (1+3i)_2] \mid i = 0, 1, \ldots, \frac{n-4}{3}\}$, and we see that $K_{n,n}$ has a hexagon covering with padding P_1 where $E(P_1) = \{((1+3i)_1, (3+3i)_2), ((2+3i)_1, (1+3i)_2), ((3+3i)_1, (2+3i)_2) \mid i = 0, 1, \ldots, \frac{n-4}{3}\}$ and so $|E(P_1)| = n$. By Lemma 3.2 Case 6, there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_2 where $|E(P_2)| = m - n + 2$. Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m + 2$.

Case 3. First, suppose $m = n + 2 \equiv 1 \pmod 6$. Then the set of hexagons $\{[1_1, 2_2, 3_1, 1_2, 2_1, 3_2], [4_1, 4_2, 6_1, 3_2, 5_1, 5_2], [3_1, 3_2, 4_1, 2_2, 7_1, 5_2], [1_1, 1_2, 6_1, 2_2, 5_1, 4_2], [1_1, 1_2, 7_1, 3_2, 6_1, 5_2], [2_1, 2_2, 4_1, 1_2, 7_1, 4_2], [2_1, 4_2, 3_1, 1_2, 5_1, 5_2]\}$ $\cup\{[1_1, (6+6i)_2, (8+6i)_1, 1_2, (9+6i)_1, 7_2], [2_1, (8+6i)_2, (10+6i)_1, 2_2, (11+6i)_1, 9_2], [1_1, (8+6i)_2, (12+6i)_1, 5_2, (9+6i)_1, 9_2], [3_1, (7+6i)_2, (10+6i)_1, 3_2, (9+6i)_1, 8_2], [3_1, (10+6i)_2, (12+6i)_1, 3_2, (13+6i)_1, 11_2], [4_1, (6+6i)_2, (9+6i)_1, 4_2, (8+6i)_1, 7_2], [4_1, (9+6i)_2, (13+6i)_1, 1_2, (11+6i)_1, 11_2], [5_1, (8+6i)_2, (11+6i)_1, 5_2, (10+6i)_1, 9_2], [6_1, (10+6i)_2, (13+6i)_1, 2_2, (12+6i)_1, 11_2], [7_1, (6+6i)_2, (13+6i)_1, 5_2, (8+6i)_1, 8_2], [7_1, (9+6i)_2, (12+6i)_1, 4_2, (11+6i)_1, 10_2], [1_1, (10+6i)_2, 5_1, (7+6i)_2, 2_1, (11+6i)_2], [2_1, (6+6i)_2, 6_1, (8+6i)_2, 4_1, (10+6i)_2], [(6+6i)_2, (10+6i)_1, 1_2, (12+6i)_1, (7+6i)_2, (11+6i)_1], [(10+6i)_2, (8+6i)_1, 2_2, (9+6i)_1, (11+6i)_2, (10+6i)_1], [3_1, (6+6i)_2, 5_1, (11+6i)_2, (8+6i)_1, (9+6i)_2]$ $[6_1, (7+6i)_2, (11+6i)_1, 3_2, (8+6i)_1, (9+6i)_2], [7_1, (7+6i)_2, (13+6i)_1, 4_2, (10+6i)_1, (11+6i)_2], [(6+6i)_2, (9+6i)_1, (10+6i)_2, (13+6i)_1, (8+6i)_2, (12+6i)_1] \mid i = 0, 1, \ldots, (m-7)/6 - 1\}$ is an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = \{(1_1, 1_2), (2_1, 4_2), (3_1, 1_2), (4_1, 2_2), (5_1, 5_2), (6_1, 3_2), (7_1, 1_2)\}$ $\cup\{((8+6i)_1, (9+6i)_2), ((9+6i)_1, (6+6i)_2), ((10+6i)_1, (11+6i)_2), ((11+6i)_1, (7+6i)_2), ((12+6i)_1, (8+6i)_2), ((13+6i)_1, (10+6i)_2) \mid i = 0, 1, \ldots, (m-7)/6 - 1\}$ and so $|E(P)| = m$. Next, suppose $m \equiv 1 \pmod 6$, $n \equiv 5 \pmod 6$, $m \geq n + 8$. Now $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{n,n}$ are $\{1_1, 2_1 \ldots, n_1\}$ and V_n, and the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n. Now $K_{n,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = n$ (by Lemma 3.3), and there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_2 satisfying $|E(P_2)| = m - n$ (by Lemma 3.2 Case 4). Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$.

Case 4. Suppose $m \equiv 3 \pmod 6$, $n \equiv 1 \pmod 6$, $m > n \geq 7$. Now $K_{m,n} = K_{n-2,n} \cup K_{m-n+2,n}$ where the partite sets of $K_{n-2,n}$ are $\{1_1, 2_1 \ldots, (n-2)_1\}$ and V_n, and the partite sets of $K_{m-n+2,n}$ are $\{(n-1)_1, n_1, \ldots, m_1\}$ and V_n. By Case 3, there is a restricted covering of $K_{n-2,n}$ with padding P_1 where $|E(P_1)| = n - 2$, and there is a restricted covering of $K_{m-n+2,n}$ with padding P_2 satisfying $|E(P_2)| = m - n + 6$ (by Lemma 3.2 Case 5). Therefore the is a restricted covering of $K_{m,n}$ with hexagons with padding

$P = P_1 \cup P_2$ where $|E(P)| = m + 4$.

Case 5. Suppose $m \equiv n \equiv 3 \pmod 6$, $m \geq n$. Now $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1 \ldots, m_1\}$ and V_n, and the partite sets of $K_{m-n,n}$ are $\{1_1, 2_1, \ldots, n_1\}$ and V_n. By Theorem 2.3, there is a decomposition of $K_{n,n} \setminus M$ where M is a perfect matching of $K_{n,n}$, say $E(M) = \{(i_1, i_2) \mid i = 1, 2, \ldots, n\}$. Taking the collection of hexagons for such a decomposition along with the set of hexagons $\{[(1+3i)_1, (3+3i)_2, (3+3i)_1, (2+3i)_2, (2+3i)_1, (1+3i)_2] \mid i = 0, 1, \ldots, \frac{n}{3}-1\}$, and we see that $K_{n,n}$ has a hexagon covering with padding P_1 where $E(P_1) = \{((1+3i)_1, (3+3i)_2), ((2+3i)_1, (1_3 i)_2), (3+3i)_1, (2+3i)) \mid i = 0, 1, \ldots, \frac{n}{3} - 1\}$ and so $|E(P_1)| = n$. By Lemma 3.2 Case 1 there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_2 where $|E(P_1)| = m - n$. Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$.

Case 6. Suppose $m \equiv 3 \pmod 6$, $n \equiv 5 \pmod 6$, $m > n$. Now $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{n,n}$ are $\{1_1, 2_1 \ldots, n_1\}$ and V_n, and the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n. Now $K_{n,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = n$ (by Lemma 3.3), and there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_2 satisfying $|E(P_2)| = m - n$ (by Lemma 3.2 Case 7). Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$.

Case 7. Suppose $m \equiv 5 \pmod 6$, $n \equiv 1 \pmod 6$, $m > n$. Now $K_{m,n} = K_{n-2,n} \cup K_{m-n+2,n}$ where the partite sets of $K_{n-2,n}$ are $\{1_1, 2_1 \ldots, (n-2)_1\}$ and V_n, and the partite sets of $K_{m-n+2,n}$ are $\{(n-1)_1, n_1, \ldots, m_1\}$ and V_n. By Case 3, there is a restricted covering of $K_{n-2,n}$ with padding P_1 where $|E(P_1)| = n - 2$, and there is a restricted covering of $K_{m-n+2,n}$ with padding P_2 satisfying $|E(P_2)| = m - n + 2$ (by Lemma 3.2 Case 1). Therefore the is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$.

Case 8. First, suppose $m = n + 2 \equiv 5 \pmod 6$. Then the set of hexagons $\{[1_1, 1_2, 2_1, 3_2, 3_1, 2_2], [1_1, 1_2, 2_1, 2_2, 5_1, 3_2], [3_1, 1_2, 5_1, 3_2, 4_1, 2_2], [3_1, 1_2, 4_1, 3_2, 5_1, 2_2]\} \cup \{[1_1, (4+6i)_2, (6+6i)_1, 1_2, (7+6i)_1, (5+6i)_2], [2_1, (4+6i)_2, (7+6i)_1, 2_2, (6+6i)_1, (5+6i)_2], [2_1, (6+6i)_2, (8+6i)_1, 2_2, (9+6i)_1, (7+6i)_2], [4_1, (6+6i)_2, (9+6i)_1, 1_2, (8+6i)_1, (7+6i)_2], [3_1, (8+6i)_2, (10+6i)_1, 3_2, (11+6i)_1, (9+6i)_2], [5_1, (8+6i)_2, (11+6i)_1, 2_2, (10+6i)_1, (9+6i)_2], [3_1, (5+6i)_2, (8+6i)_1, 3_2, (7+6i)_1, (6+6i)_2], [4_1, (8+6i)_2, (9+6i)_1, (4+6i)_2, (8+6i)_2, (9+6i)_2], [3_1, (4+6i)_2, 4_1, (5+6i)_2, 5_1, (7+6i)_2], [(11+6i)_1, (4+6i)_2, (10+6i)_1, (6+6i)_2, (6+6i)_1, (7+6i)_2], [1_1, (8+6i)_2, (6+6i)_1, 3_2, (9+6i)_1, (9+6i)_2], [1_1, (6+6i)_2, (11+6i)_1, 1_2, (10+6i)_1, (7+6i)_2], [2_1, (8+6i)_2, (7+6i)_1, (7+6i)_2, (6+6i)_1, (9+6i)_2], [5_1, (4+6i)_2, (9+6i)_1, (5+6i)_2, (10+6i)_1, (6+6i)_2], [(11+6i)_1, (5+6i)_2, (8+6i)_1, (8+6i)_2, (7+6i)_1, (9+6i)_2] \mid i = 0, 1, \ldots, (m-5)/6 - 1\}$ is an unrestricted covering of $K_{m,n}$

with padding P where $E(P) = \{(3_1, 2_2), (1_1, 1_2), (2_1, 1_2), (3_1, 1_2), (3_1, 2_2), (4_1, 3_2), (5_1, 2_2), (5_1, 3_2), (5_1, 3_2)\} \cup \{((6+6i)_1, (7+6i)_2), ((7+6i)_1, (8+6i)_2), ((8+6i)_1, (5+6i)_2), ((9+6i)_1, (4+6i)_2), ((10+6i)_1, (6+6i)_2), ((11+6i)_1, (9+6i)_2) \mid i = 0, 1, \ldots, (m-5)/6 - 1\}$ and so $|E(P)| = m + 4$.

Next, suppose $m \equiv 5 \pmod 6$, $n \equiv 3 \pmod 6$, $m \geq n$. Now $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{n,n}$ are $\{1_1, 2_1 \ldots, n_1\}$ and V_n, and the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n. By Theorem 2.3, there is a decomposition of $K_{n,n} \setminus M$ where M is a perfect matching of $K_{n,n}$, say $E(M) = \{(i_1, 1_2) \mid i = 1, 2, \ldots, n\}$. Taking the collection of hexagons for such a decomposition along with the set of hexagons $\{[(1+3i)_1, (3+3i)_2, (3+3i)_1, (2+3i)_2, (2+3i)_1, (1+3i)_2] \mid i = 0, 1, \ldots, \frac{n}{3}-1\}$, and we see that $K_{n,n}$ has a hexagon covering with padding P_1 where $E(P_1) = \{((1+3i)_1, (3+3i)_2), ((2+3i)_1, (1+3i)_2), (3+3i)_1, (2+3i)) \mid i = 0, 1, \ldots, \frac{n}{3} - 1\}$ and so $|E(P_1)| = n$. By Lemma 3.2 Case 3 there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_2 where $|E(P_1)| = m - n + 4$. Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m + 4$.

Case 9. Suppose $m \equiv n \equiv 5 \pmod 6$, $m > n$. Now $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{n,n}$ are $\{1_1, 2_1 \ldots, n_1\}$ and V_n, and the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n. Now $K_{n,n}$ has a restricted hexagon covering with padding P_1 where $|E(P_1)| = n$ (by Lemma 3.3), and there is a restricted hexagon covering of $K_{m-n,n}$ with padding P_2 satisfying $|E(P_2)| = m - n$ (by Lemma 3.2 Case 1). Therefore there is a restricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m$. ∎

Theorem 3.1 *A minimal restricted hexagon covering of $K_{m,n}$ (where $m \geq 3$ and $n \geq 3$) with padding P satisfies*
(1) when $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$, $|E(P)| = m + k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m+k) \equiv 0 \pmod 6$,
(2) when $m \equiv n \equiv 1 \pmod 2$ and $m \geq n$, $|E(P)| = m + k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m+k) \equiv 0 \pmod 6$,
(3) when $m \equiv 0 \pmod 6$ and $n \equiv 0 \pmod 2$, $|E(L)| = 0$,
(4) when $m \equiv n \equiv 2 \pmod 6$ or $m \equiv n \equiv 4 \pmod 6$, then $|E(P)| = 2$, and
(5) when $m \equiv 2 \pmod 6$ and $n \equiv 4 \pmod 6$, then $|E(P)| = 4$.

4. Unrestricted Hexagon Coverings of $K_{m,n}$

The following two result immediately show a difference between a restricted hexagon covering (which does not exist for $K_{1,n}$ or $K_{2,n}$) and an unrestricted hexagon covering (which the result describes).

Lemma 4.1 *A minimal unrestricted covering of $K_{1,n}$, $n \geq 5$, has a padding P where $|E(P)| = 2n$ when n is even and $|E(P)| = 2n + 3$ when n is odd.*

Proof. For n even, $n \geq 6$, we have V_1 and V_n as the partite sets of $K_{2,n}$. If a hexagon in such a covering contains no vertices of V_1, then it must contain 6 edges in the padding. If a hexagon in a covering contains 1 vertex of V_1, then it must contain at least 4 edges in P and at most 2 edges in $K_{2,n}$. Since $K1, n$ contains n edges, then an unrestricted covering with padding P must satisfy $|E(P)| \geq 2n$. The set of hexagons $\{[1_1, 1_2, 5_2, 4_2, 3_2, 2_2], [1_1, 3_2, 2_2, 1_2, 5_2, 4_2], [1_1, 5_2, 4_2, 3_2, 2_2, 6_2]\} \cup \{[1_1, (5+2i)_2, 3_2, 2_2, 1_2, (6+2i)_2] \mid i = 1, 2, \ldots, (n-6)/2$ forms an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = \{(1_2, 5_2), (4_2, 5_2), (3_2, 4_2), (2_2, 3_2), (2_2, 3_2), (1_2, 3_2), (1_2, 5_2), (4_2, 5_2), (4_2, 5_2), (3_2, 4_2), (2_2, 3_2), (2_2, 6_2)\} \cup \{(3_2, (5+2i)_2), (2_2, 3_2), (1_2, 2_2), (1_2, (6+2i)_2) \mid i = 1, 2, \ldots, (n-6)/2\}$ and so $|E(P)| = 2n$.

For n odd, as when n is even, each hexagon of an unrestricted covering contains at least 4 edges in the padding, so an unrestricted with padding P must satisfy $|E(P)| \geq 2n$. Since $|E(K_{m,n})| + |E(P)| \equiv 0 \pmod 6$, it follows that $|E(P)| \geq 2n+3$. The set of hexagons $\{[1_1, 1_2, 5_2, 4_2, 3_2, 2_2], [1_1, 3_2, 2_2, 1_2, 5_2, 4_2], [1_1, 4_2, 1_2, 2_2, 3_2, 5_2]\} \cup \{[(1_1, (4+2i)_2, 3_2, 2_2, 1_2, (5+2i)_2] \mid i = 1, 2, \ldots, (n-5)/2\}$ forms an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = \{(1_2, 5_2), (4_2, 5_2), (3_2, 4_2), (2_2, 3_2), (2_2, 3_2), (1_2, 3_2), (1_2, 5_2), (4_2, 5_2), (1_1, 4_2), (1_2, 4_2), (1_2, 2_2), (2_2, 3_2), (3_2, 5_2)\} \cup \{(3_2, (4+2i)_2), (2_2, 3_2), (1_2, 2_2), (1_2, (5+2i)_2) \mid i = 1, 2, \ldots, (n-5)/2\}$ and so $|E(P)| = 2n+3$. ■

Lemma 4.2 *A minimal unrestricted covering of $K_{2,n}$, $n \geq 4$, has a padding P where $|E(P)| = n$ when n is even and $|E(P)| = n+3$ when n is odd.*

Proof. For n even, $n \geq 4$, we have V_2 and V_n as the partite sets of $K_{2,n}$. If a hexagon in such a covering contains no vertices of V_2, then it must contain 6 edges in the padding. If a hexagon in a covering contains 1 vertex of V_2, then it must contain at least 4 edges in P and at most 2 edges in $K_{2,n}$. If a hexagon in a covering contains 2 vertices of V_2, then it must contain either (1) at least 2 edges in P and at most 4 edges in $K_{2,n}$, or (2) at least 4 edges in P and at most 2 edges in $K_{2,n}$. Since $K_{2,n}$ contains $2n$ edges, then an unrestricted covering with padding P must satisfy $|E(P)| \geq n$. Since $n \geq 4$ is even, then $n = 4n_1 + 6n_2$ for some $n_1, n_2 \in \mathbb{N}$. Then the set of hexagons: $\{[1_1, (1+4i)_2, (4+4i)_2, (3+4i)_2, 2_1, (2+4i)_2], [1_1, (3+4i)_2, (2+4i)_2, (1+4i)_2, 2_1, (4+4i)_2] \mid i = 0, 1, \ldots, n_1 - 1\} \cup \{[1_1, (6+6j)_2, (2+6j)_2, (1+6j)_2, 2_1, (5+6j)_2], [1_1, (1+6j)_2, (4+6j)_2, (3+6j)_2, 2_1, (2+6j)_2], [1_1, (3+6j)_2, (2+6j)_2, (6+6j)_2, 2_1, (4+6j)_2] \mid j = 0, 1, \ldots, n_2 - 1\}$ is an unrestricted covering of $K_{2,n}$ with padding $P = \{((1+4i)_2, (2+4i)_2), ((2+4i)_2, (3+4i)_2), ((3+4i)_2, (4+4i)_2), ((1+4i)_2, (4+4i)_2) \mid i = 0, 1, \ldots, n_1 - 1\} \cup \{((1+6j)_2, (2+6j)_2), ((2+6j)_2, (3+6j)_2), ((3+6j)_2, (4+6j)_2), ((1+6j)_2, (4+6j)_2), 2 \times ((2+6j)_2, (6+6j)_2) \mid j = 0, 1, \ldots, n_2 - 1\}$ and so $|E(P)| = 4n_1 + 6n_2 = n$ and this unrestricted covering is minimal.

For n odd, as when n is even, each hexagon of a covering of $K_{2,n}$ contains at least 2 edges of the padding and at most 4 edges of $K_{2,n}$. Since $|E(K_{2,n})| = 2n$, then the number of hexagons in a covering must be at least $\lceil 2n/4 \rceil = \lceil n/2 \rceil = (n+1)/2$ since n is odd. Since each hexagon contains at least 2 edges of the padding P, we have $|E(P)| \geq n+1$. Now we need $|E(K_{2,n})| + |E(P)| \equiv 0 \pmod 6$ and $|E(K_{2,n})| + |E(P)| \geq 3n+1$, so $|E(P)| \geq n+3$. Since n is odd and $n \geq 5$, then either (1) $n = 4\ell+5$ where $\ell = (n-5)/4 \in \mathbb{N}$, or (2) $n = 4\ell+7$ where $\ell = (n-7)/4 \in \mathbb{N}$. Define $A = \{[1_1, (1+4i)_2, (4+4i)_2, (3+4i)_2, 2_1, (2+4i)_2], [1_1, (3+4i)_2, (2+4i)_2, (1+4i)_2, 2_1, (4+4i)_2] \mid i = 0, 1, \ldots, \ell-1\}$. For $n = 4\ell+5$, consider the set of blocks $A \cup \{[1_1, (n-4)_2, 2_1, n_2, (n-1)_2, (n-2)_2], [1_1, (n-4)_2, 2_1, (n-1)_2, (n-2)_2, (n-3)_2], [1_1, (n-1)_2, (n-2)_2, 2_1, (n-3)_2, n_2]\}$. This is an unrestricted covering of $K_{2,n}$ with padding P where $E(P) = \{((1+4i)_2, (2+4i)_2), ((2+4i)_2, (3+4i)_2), ((3+4i)_2, (4+4i)_2), ((1+4i)_2, (4+4i)_2) \mid i = 0, 1, \ldots, \ell-1\} \cup \{(1_1, (n-4)_2), (2_1, (n-4)_2), ((n-3)_2, (n-2)_2), ((n-3)_2, n_2), 3 \times ((n-2)_2, (n-1)_2), ((n-1)_2, n_2)\}$. Since $|E(P)| = 4\ell+8 = n+3$, the covering is minimal. For $n = 4\ell+7$, consider the set of blocks $A \cup \{[1_1, (n-6)_2, 2_1, (n-2)_2, (n-3)_2, (n-4)_2], [1_1, (n-6)_2, 2_1, (n-3)_2, (n-4)_2, (n-5)_2], [1_1, (n-1)_2, (n-4)_2, 2_1, (n-5)_2, n_2], [1_1, (n-3)_2, n_2, 2_1, (n-1)_2, (n-2)_2]\}$. This is an unrestricted covering of $K_{2,n}$ with padding $P = \{((1+4i)_2, (2+4i)_2), ((2+4i)_2, (3+4i)_2), ((3+4i)_2, (4+4i)_2), ((1+4i)_2, (4+4i)_2) \mid i = 0, 1, \ldots, \ell-1\} \cup \{(1_1, (n-6)_2), (2_1, (n-6)_2), ((n-5)_2, (n-4)_2), ((n-5)_2, n_2), 2 \times ((n-4)_2, (n-3)_2), ((n-4)_2, (n-1)_2), ((n-3)_2, n_2), ((n-3)_2, (n-2)_2), ((n-2)_2, (n-1)_2)\}$. Since $|E(P)| = 4\ell+10 = n+3$, the covering is minimal. ■

When m and n are both even, the constructions of Lemma 3.1 immediately give the following.

Lemma 4.3 *A minimal unrestricted hexagon covering of $K_{m,n}$ where m and n are even, $m \geq 4$ and $n \geq 4$, has a padding P satisfying:*
(1) $|E(P)| = 0$ when $m \equiv 0 \pmod 6$,
(2) $|E(P)| = 2$ when $m \equiv n \equiv 2 \pmod 6$ or $m \equiv n \equiv 4 \pmod 6$, and
(3) $|E(P)| = 4$ when $m \equiv 2 \pmod 6$ and $n \equiv 4 \pmod 6$.

Lemma 4.4 *A minimal unrestricted hexagon covering of $K_{m,n}$ where m is even, $m \geq 4$, and n is odd, $n \geq 3$, has a padding P satisfying $|E(P)| = m/2+k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m/2+k) \equiv 0 \pmod 6$.*

Proof. Since each vertex of V_m is of odd degree in $K_{m,n}$, in the padding of a covering each of these vertices will be of odd degree. Therefore in a restricted covering of $K_{m,n}$ with padding P, it is necessary that $|E(P)| \geq m/2$. Since a covering yields a decomposition of $K_{m,n} \cup P$, then it is necessary that $|E(K_{m,n})| + |E(P)| \equiv 0 \pmod 6$.

Case 1. First, suppose $m \equiv 0 \pmod{12}$ and $n = 3$. Then the set of hexagons $\{[(1+12i)_1, 2_2, (3+12i)_1, 1_2, (2+12i)_1, 3_2], [(4+12i)_1, 2_2, (6+12i)_1, 1_2, (5+12i)_1, 3_2], [(7+12i)_1, 2_2, (9+12i)_1, 1_2, (8+12i)_1, 3_2], [(10+12i)_1, 2_2, (12+12i)_1, 1_2, (11+12i)_1, 3_2], [(1+12i)_1, 1_2, (4+12i)_1, (5+12i)_1, 2_2, (2+12i)_1], [(6+12i)_1, 3_2, (9+12i)_1, (10+12i)_1, 1_2, (7+12i)_1], [(3+12i)_1, 3_2, (12+12i)_1, (11+12i)_1, 2_2, (8+12i)_1] \mid i = 0, 1, \ldots, m/12-1\}$ is an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = \{((1+12i)_1, (2+12i)_1), ((3+12i)_1, (8+12i)_1), ((4+12i)_1, (5+12i)_1), ((6+12i)_1, (7+12i)_1), ((9+12i)_1, (10+12i)_1), ((11+12i)_1, (12+12i)_1) \mid = 0, 1, \ldots, m/12-1\}$ and so $|E(P)| = m/2$.

Second, suppose $m \equiv 0 \pmod{12}$ and $n = 5$. Then the set of hexagons $\{[(1+12i)_1, 1_2, (2+12i)_1, 2_2, (3+12i)_1, 3_2], [(1+12i)_1, 4_2, (3+12i)_1, 1_2, (5+12i)_1, 5_2], [(2+12i)_1, 3_2, (6+12i)_1, 2_2, (4+12i)_1, 4_2], [(4+12i)_1, 3_2, (5+12i)_1, 4_2, (6+12i)_1, 5_2], [(7+12i)_1, 4_2, (8+12i)_1, 5_2, (9+12i)_1, 1_2], [(7+12i)_1, 2_2, (9+12i)_1, 4_2, (11+12i)_1, 3_2], [(8+12i)_1, 1_2, (12+12i)_1, 5_2, (10+12i)_1, 2_2], [(10+12i)_1, 1_2, (11_1+12i)_1, 2_2, (12+12i)_1, 3_2], [(1+12i)_1, 2_2, (5+12i)_1, (7+12i)_1, 5_2, (11+12i)_1], [(2+12i)_1, 5_2, (3+12i)_1, (10+12i)_1, 4_2, (12+12i)_1], [(4+12i)_1, 1_2, (6+12i)_1, (8+12i)_1, 3_2, (9+12i)_1] \mid i = 0, 1, \ldots, m/12-1\}$ is an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = \{((1+12i)_1, (11+12i)_1), ((2+12i)_1, (12+12i)_1), ((3+12i)_1, (10+12i)_1), ((4+12i)_1, (9+12i)_1), ((5+12i)_1, (7+12i)_1), ((6+12i)_1, (8+12i)_1) \mid i = 0, 1, \ldots, m/12-1\}$ and so $|E(P)| = m/2$.

Finally suppose $m \equiv 0 \pmod{12}$, $n \equiv 1 \pmod 2$, and $n > 5$. Now $K_{m,n} \subset K_{m,n-3} \cup \frac{m}{3} \times C_6 \cup \frac{m}{4} \times C_6$ where the partite sets of $K_{m,n-3}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2\}$, $\frac{m}{3} \times C_6 = \{[(1+3i)_1, 2_2, (3+3i)_1, 1_2, (2+3i)_1, 3_2] \mid i = 0, 1, \ldots, \frac{m}{3}-1\}$, and $\frac{m}{4} \times C_6 = \{[(1+12i)_1, 1_2, (4+12i)_1, (2+12i)_1, 2_2, (5+12i)_1], [(3+12i)_1, 3_2, (6+12i)_1, (7+12i)_1, 1_2, (10+12i)_1], [(8+12i)_1, 2_2, (11+12i)_1, (12+12i)_1, 3_2, (9+12i)_1] \mid i = 0, 1, \ldots, \frac{m}{4}-1\}$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding P where $E(P) = \{((1+12i)_1, (5+12i)_1), ((2+12i)_1, (4+12i)_1), ((3+12i)_1, (10+12i)_1), ((6+12i)_1, (7+12i)_1), ((8+12i)_1, (9+12i)_1), ((11+12i)_1, (12+12i)_1) \mid i = 0, 1, \ldots, m/12-1\}$ and so $|E(P)| = \frac{m}{2}$ and the unrestricted covering is minimal.

Case 2. Suppose $m \equiv 2 \pmod{12}$, $m \geq 14$, and $n \equiv 1 \pmod 6$, $n \geq 7$. Now $K_{m,n} = K_{14,7} \cup K_{m-14,7} \cup K_{m,n-7}$ where the partite sets of $K_{14,7}$ are $\{1_1, 2_1, \ldots, 14_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$, the partite sets of $K_{m-14,7}$ are $\{15_1, 16_1, \ldots, m_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of $K_{m,n-7}$ are V_m and $\{8_2, 9_2, \ldots, n_2\}$. There exists an unrestricted covering of $K_{m-14,7}$ with padding P_1 where $|E(P_1)| = (m-14)/2$ by Case 1 and there exists a hexagon decomposition of $K_{m,n-7}$ by Theorem 2.2. Next, we note that $K_{14,7} = K_{7,7} \cup K_{7,7}$ where the partite sets of the first copy of $K_{7,7}$ are $\{1_1, 2_1, \ldots, 7_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of the second copy of $K_{7,7}$ are $\{8_1, 9_1, \ldots, 14_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$. By Theorem 2.3, there is a hexagon decomposition of $K_{7,7} \setminus M$ where M is a matching of $K_{7,7}$. So there is a hexagon decomposition of $K_{14,7} \setminus M_1$ where $E(M_1) = \{(i_1, i_2), ((i+7)_1, i_2) \mid i = 1, 2, \ldots, 7\}$. This decomposition along with the set $\{[1_1, 1_2, 8_1, 9_1, 2_2, 2_1], [3_1, 3_2, 10_1, 11_1, 4_2, 4_1], [5_1, 5_2, 12_1, 13_1, 6_2, 6_1], [6_1, 6_2, 13_1, 14_1, 7_2, 7_1]\}$ forms an unrestricted covering of $K_{14,7}$ with padding P_2 where $E(P_2) = \{(1_1, 2_1), (3_1, 4_1), (5_1, 6_1), (6_1, 7_1), (6_1, 6_2), (8_1, 9_1), (10_1, 11_1), (12_1, 13_1), (13_1, 14_1), (13_1, 6_2)\}$ and so $|E(P_2)| = 10$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 3$.

Case 3. Suppose $m \equiv 2 \pmod{12}$, $m \geq 14$, and $n \equiv 3 \pmod 6$. Now $K_{m,n} = K_{14,3} \cup K_{m-14,3} \cup K_{m,n-3}$ where the partite sets of $K_{14,3}$ are $\{1_1, 2_1, \ldots, 14_1\}$ and $\{1_2, 2_2, 3_2\}$, the partite sets of $K_{m-14,3}$ are $V_m \setminus \{1_1, 2_1, \ldots, 14_1\}$ and $\{1_2, 2_2, 3_2\}$, and the partite sets of $K_{m,n-3}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2\}$. There exists an unrestricted covering of $K_{m-14,3}$ with padding P_1 where $|E(P_1)| = (m-14)/2$ by Case 1 and $K_{m,n-3}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{14,3}$, namely the set $\{[1_1, 2_2, 2_1, 11_1, 3_2, 14_1], [3_1, 1_2, 6_1, 7_1, 2_2, 4_1], [9_1, 1_2, 12_1, 13_1, 2_2, 10_1,] [3_1, 2_2, 5_1, 1_2, 4_1, 3_2], [6_1, 2_2, 8_1, 1_2, 7_1, 3_2], [9_1, 2_2, 11_1, 1_2, 10_1, 3_2], [12_1, 2_2, 14_1, 1_2, 13_1, 3_2], [1_1, 1_2, 2_1, 5_1, 3_2, 8_1], [1_1, 1_2, 3_1, 2_2, 2_1, 3_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 14_1), (2_1, 11_1), (3_1, 4_1), (6_1, 7_1), (9_1, 10_1), (12_1, 13_1), (1_1, 1_2), (1_1, 8_1), (2_1, 2_2), (2_1, 5_1), (3_1, 1_2), (3_1, 2_2)\}$ and so $|E(P_2)| = 12$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 5$.

Case 4. Suppose $m \equiv 2 \pmod{12}$, $m \geq 14$, and $n \equiv 5 \pmod 6$, $n \geq 5$. Now $K_{m,n} = K_{14,5} \cup K_{m-14,5} \cup K_{m,n-5}$ where the partite sets of $K_{14,5}$ are $\{1_1, 2_1, \ldots, 14_1\}$ and $\{1_2, 2_2, 3_2, 4_2, 5_2\}$, the partite sets of $K_{m-14,5}$ are $\{15_1, 16_1, \ldots, m_1\}$ and $\{1_2, 2_2, 3_2, 4_2, 5_2\}$, and the partite sets of $K_{m,n-5}$ are V_m and $\{6_2, 7_2, \ldots, n_2\}$. There exists an unrestricted covering of $K_{m-14,5}$ with padding P_1 where $|E(P_1)| = (m-14)/2$ by Case 1 and $K_{m,n-5}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{14,5}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [2_1, 3_2, 4_1, 5_2, 3_1, 4_2], [5_1, 1_2, 6_1, 2_2, 7_1, 3_2], [6_1, 3_2, 8_1, 5_2, 7_1, 4_2], [8_1, 1_2, 4_1, 2_2, 1_1, 4_2], [9_1, 2_2, 12_1, 4_2, 10_1, 3_2], [11_1, 5_2, 13_1, 4_2, 14_1, 1_2], [9_1, 5_2, 14_1, 2_2, 11_1, 4_2], [10_1, 5_2, 12_1, 3_2, 13_1, 1_2], [1_1, 5_2, 2_1, 9_1, 1_2, 12_1], [3_1, 1_2, 7_1, 5_1, 2_2, 8_1], [10_1, 2_2, 13_1, 5_1, 5_2, 6_1], [11_1, 3_2, 14_1, 4_1, 4_2, 5_1]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 12_1), (2_1, 9_1), (3_1, 8_1), (4_1, 14_1), (5_1, 7_1), (5_1, 11_1), (5_1, 13_1), (6_1, 10_1)\}$ and $|E(P_2)| = 8$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 1$.

Case 5. Suppose $m \equiv 4 \pmod{12}$, and $n \equiv 1 \pmod 6$, $n \geq 7$. Now

$K_{m,n} = K_{m-4,n} \cup K_{4,n-7} \cup K_{4,7}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-7}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of $K_{4,7}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$. There exists an unrestricted covering of $K_{m-4,n}$ with padding P_1 where $|E(P_1)| = (m-4)/2$ by Case 1 and $K_{4,n-7}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{4,7}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [2_1, 5_2, 4_1, 7_2, 3_1, 6_2], [1_1, 6_2, 4_1, 4_2, 2_1, 7_2], [1_1, 2_2, 4_1, 1_2, 3_1, 5_2], [1_1, 4_2, 3_1, 4_1, 3_2, 2_1]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 2_1), (3_1, 4_1)\}$ and so $|E(P_2)| = 2$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2$.

Case 6. Suppose $m \equiv 4 \pmod{12}$, $n \equiv 3 \pmod 6$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-3} \cup K_{4,3}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-3}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2\}$, and the partite sets of $K_{4,3}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, 3_2\}$. There exists an unrestricted covering of $K_{m-4,n}$ with padding P_1 where $|E(P_1)| = (m-4)/2$ by Case 1 and $K_{4,n-3}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{4,3}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [1_1, 2_2, 4_1, 1_2, 3_1, 2_1], [1_1, 1_2, 2_1, 3_2, 4_1, 2_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 2_1), (2_1, 3_1), (1_1, 1_2), (1_1, 2_2), (2_1, 1_2), (4_1, 2_2)\}$ and so $|E(P_2)| = 6$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 4$.

Case 7. First, suppose $m \equiv 4 \pmod{12}$ and $n = 5$. Now $K_{m,n} = K_{m-4,5} \cup K_{4,5}$ where the partite sets of $K_{m-4,5}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, and the partite sets of $K_{4,5}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and V_n. There is an unrestricted covering of $K_{m-4,5}$ by Case 1 with padding P_1 where $|E(P_1)| = (m-4)/2$. Next, we note that there is an unrestricted hexagon covering of $K_{4,5}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [2_1, 1_2, 3_1, 4_2, 4_1, 5_2], [1_1, 2_2, 4_1, 3_2, 2_1, 4_2], [1_1, 1_2, 4_1, 4_2, 3_1, 5_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 1_2), (2_1, 1_2), (3_1, 4_2), (4_1, 4_2)\}$ and so $|E(P_2)| = 4$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 2$.

Now suppose $m \equiv 4 \pmod{12}$ and $n \equiv 5 \pmod 6$, $n \geq 11$. Now $K_{m,n} = K_{m,n-4} \cup K_{m,4}$ where the partite sets of $K_{m,n-4}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2, 4_2\}$ and the partite sets of $K_{m,4}$ are V_m and $\{1_2, 2_2, 3_2, 4_2\}$. There exists an unrestricted hexagon covering of $K_{m,n-4}$ with padding P_1 where $|E(P_1)| = m/2$ by Case 5 and there is a restricted hexagon covering of $K_{m,4}$ with padding P_2 where $|E(P_2)| = 2$ by Lemma 3.1 Case 3. Therefore there is an unrestricted hexagon covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 2$.

Case 8. First, suppose $m \equiv 6 \pmod{12}$ and $n = 3$. Now $K_{m,n} = K_{6,3} \cup K_{m-6,3}$ where the partite sets of $K_{6,3}$ are $\{1_1, 2_1, \ldots, 6_1\}$ and V_n, and the partite sets of $K_{m-6,3}$ are $V_m \setminus \{1_1, 2_1, \ldots, 6_1\}$ and V_n. There exists an unrestricted covering of $K_{m-6,3}$ with padding P_1 where $|E(P_1)| = (m-6)/2$

by Case 1. Next, we note that there is an unrestricted hexagon covering of $K_{6,3}$, namely the set $\{[1_1, 2_2, 3_1, 1_2, 2_1, 3_2], [4_1, 2_2, 6_1, 1_2, 5_1, 3_2], [1_1, 1_2, 4_1, 5_1, 2_2, 2_1], [3_1, 1_2, 5_1, 2_2, 6_1, 3_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 2_1), (4_1, 5_1), (3_1, 1_2), (5_1, 1_2), (5_1, 2_2), (6_1, 2_2)\}$ and so $|E(P_2)| = 6$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 3$.

Second, suppose $m \equiv 6 \pmod{12}$ and $n = 5$. Now $K_{m,n} = K_{6,5} \cup K_{m-6,5}$ where the partite sets of $K_{6,5}$ are $\{1_1, 2_1, \ldots, 6_1\}$ and V_n, and the partite sets of $K_{m-6,5}$ are $V_m \setminus \{1_1, 2_1, \ldots, 6_1\}$ and V_n. There exists an unrestricted covering of $K_{m-6,n}$ with padding P_1 where $|E(P_1)| = (m-6)/2$ by Case 1. Next, we note that there is an unrestricted hexagon covering of $K_{6,5}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 3_1, 3_2], [1_1, 4_2, 3_1, 1_2, 5_1, 5_2], [2_1, 3_2, 6_1, 2_2, 4_1, 4_2], [4_1, 3_2, 5_1, 4_2, 6_1, 5_2], [1_1, 2_2, 5_1, 3_1, 5_2, 2_1], [4_1, 1_2, 6_1, 5_2, 5_1, 4_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 2_1), (3_1, 5_1), (4_1, 4_2), (5_1, 4_2), (5_1, 5_2), (6_1, 5_2)\}$ and so $|E(P_2)| = 6$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 3$.

Finally, suppose $m \equiv 6 \pmod{12}$, $n \equiv 1 \pmod{2}$, and $n > 5$. Now $K_{m,n} = K_{6,n} \cup K_{m-6,n}$ where the partite sets of $K_{6,n}$ are $\{1_1, 2_1 \ldots, 6_1\}$ and V_n, and the partite sets of $K_{m-6,n}$ are $V_m \setminus \{1_1, 2_1, \ldots, 6_1\}$ and V_n. There exists a restricted covering of $K_{6,n}$ with padding P_1 where $|E(P_1)| = 6$ by Lemma 3.2 Case 1 and there exists an unrestricted covering of $K_{m-6,n}$ with padding P_2 where $|E(P_2)| = (m-6)/2$ by Case 1. Therefore there is a unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 3$ and the unrestricted covering is minimal.

Case 9. Suppose $m \equiv 8 \pmod{12}$, $n \equiv 1 \pmod{6}$, and $n \geq 7$. Now $K_{m,n} = K_{8,7} \cup K_{m-8,7} \cup K_{m,n-7}$ where the partite sets of $K_{8,7}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$, the partite sets of $K_{m-8,7}$ are $V_m \setminus \{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of $K_{m,n-7}$ are V_m and $V_n \setminus \{1_2, 2_2, \ldots, 7_2\}$. There exists an unrestricted covering of $K_{m-8,7}$ with padding P_1 where $|E(P_1)| = (m-8)/2$ by Case 1 and $K_{m,n-7}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{8,7}$, namely the set $\{[1_1, 2_2, 6_1, 5_2, 2_1, 3_2], [1_1, 4_2, 5_1, 1_2, 4_1, 6_2], [3_1, 3_2, 5_1, 2_2, 4_1, 4_2], [2_1, 6_2, 5_1, 5_2, 4_1, 7_2], [6_1, 3_2, 8_1, 2_2, 7_1, 4_2], [6_1, 6_2, 8_1, 5_2, 7_1, 7_2], [1_1, 1_2, 8_1, 7_2, 3_1, 5_2], [2_1, 1_2, 7_1, 6_2, 3_1, 2_2], [1_1, 7_2, 5_1, 8_1, 4_2, 2_1], [3_1, 1_2, 6_1, 7_1, 3_2, 4_1]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 2_1), (3_1, 4_1), (5_1, 8_1), (6_1, 7_1)\}$ and so $|E(P_2)| = 4$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2$.

Case 10. Suppose $m \equiv 8 \pmod{12}$, $n \equiv 3 \pmod{6}$. Now $K_{m,n} = K_{8,3} \cup K_{m-8,n} \cup K_{8,n-3}$ where the partite sets of $K_{8,3}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, 3_2\}$, the partite sets of $K_{m-8,n}$ are $V_m \setminus \{1_1, 2_1, \ldots, 8_1\}$ and V_n, and the partite sets of $K_{8,n-3}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $V_n \setminus \{1_2, 2_2, 3_2\}$. There exists an unrestricted covering of $K_{m-8,n}$ with padding P_1 where $|E(P_1)| =$

$(m-8)/2$ by Case 1 and $K_{8,n-3}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{8,3}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 4_1, 3_2], [1_1, 2_2, 7_1, 5_1, 3_2, 2_1], [3_1, 1_2, 6_1, 8_1, 3_2, 4_1], [3_1, 2_2, 5_1, 1_2, 4_1, 3_2], [6_1, 2_2, 8_1, 1_2, 7_1, 3_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 2_1), (3_1, 4_1), (5_1, 7_1), (6_1, 8_1), (4_1, 3_2), (4_1, 3_2)\}$ and so $|E(P_2)| = 6$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 2$.

Case 11. Suppose $m \equiv 8 \pmod{12}$, $n \equiv 5 \pmod 6$. Now $K_{m,n} = K_{8,5} \cup K_{m-8,5} \cup K_{m,n-5}$ where the partite sets of $K_{8,5}$ are $\{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 5_2\}$, the partite sets of $K_{m-8,5}$ are $V_m \setminus \{1_1, 2_1, \ldots, 8_1\}$ and $\{1_2, 2_2, \ldots, 5_2\}$, and the partite sets of $K_{m,n-5}$ are V_m and $V_n \setminus \{1_2, 2_2, \ldots, 5_2\}$. There exists an unrestricted covering of $K_{m-8,5}$ with padding P_1 where $|E(P_1)| = (m-8)/2$ by Case 1 and $K_{m,n-5}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{8,5}$, namely the set $\{[1_1, 1_2, 3_1, 3_2, 6_1, 2_2], [2_1, 1_2, 4_1, 5_1, 2_2, 3_1], [3_1, 4_2, 5_1, 3_2, 4_1, 5_2], [6_1, 4_2, 8_1, 3_2, 7_1, 5_2], [1_1, 3_2, 2_1, 2_2, 8_1, 5_2], [2_1, 4_2, 7_1, 1_2, 5_1, 5_2], [4_1, 2_2, 7_1, 8_1, 1_2, 6_1], [1_1, 1_2, 2_1, 2_2, 4_1, 4_2]\}$ with padding P_2 satisfying $E(P_2) = \{(2_1, 3_1), (4_1, 5_1), (1_1, 1_2), (2_1, 1_2), (2_1, 2_2), (4_1, 2_2), (4_1, 6_1), (7_1, 8_1)\}$ and so $|E(P_2)| = 8$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 4$.

Case 12. Suppose $m \equiv 10 \pmod{12}$ and $n \equiv 1 \pmod 6$, $n \geq 7$. Now $K_{m,n} = K_{m-4,n} \cup K_{4,n-7} \cup K_{4,7}$ where the partite sets of $K_{m-4,n}$ are $V_m \setminus \{1_1, 2_1, 3_1, 4_1\}$ and V_n, the partite sets of $K_{4,n-7}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $V_n \setminus \{1_2, 2_2, \ldots, 7_2\}$, and the partite sets of $K_{4,7}$ are $\{1_1, 2_1, 3_1, 4_1\}$ and $\{1_2, 2_2, \ldots, 7_2\}$. There exists an unrestricted covering of $K_{m-4,n}$ with padding P_1 where $|E(P_1)| = m/2 + 1$ by Case 8 and $K_{4,n-7}$ can be decomposed by Theorem 2.2. Then by Case 5 gives an unrestricted hexagon covering of $K_{4,7}$ with padding P_2 where $|E(P_2)| = 2$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 3$.

Case 13. Suppose $m \equiv 10 \pmod{12}$, $n \equiv 3 \pmod 6$. Now $K_{m,n} = K_{10,3} \cup K_{m-10,3} \cup K_{m,n-3}$ where the partite sets of $K_{10,3}$ are $\{1_1, 2_1, \ldots, 10_1\}$ and $\{1_2, 2_2, 3_2\}$, the partite sets of $K_{m-10,3}$ are $V_m \setminus \{1_1, 2_1, \ldots, 10_1\}$ and $\{1_2, 2_2, 3_2\}$, and the partite sets of $K_{m,n-3}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2\}$. There exists an unrestricted hexagon covering of $K_{m-10,3}$ with padding P_1 where $|E(P_1)| = (m-10)/2$ by Case 1 and $K_{m,n-3}$ can be decomposed by Theorem 2.2. Next, we note that there is an unrestricted hexagon covering of $K_{10,3}$, namely the set $\{[1_1, 1_2, 2_1, 2_2, 10_1, 3_2], [1_1, 2_2, 3_1, 5_1, 1_2, 8_1], [4_1, 3_2, 7_1, 9_1, 2_2, 6_1], [2_1, 2_2, 4_1, 1_2, 3_1, 3_2], [5_1, 2_2, 7_1, 1_2, 6_1, 3_2], [8_1, 2_2, 10_1, 1_2, 9_1, 3_2]\}$ with padding P_2 satisfying $E(P_2) = \{(1_1, 8_1), (2_1, 2_2), (3_1, 5_1), (4_1, 6_1), (7_1, 9_1), (10_1, 2_2)\}$ and so $|E(P_2)| = 6$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 1$.

Case 14. Suppose $m \equiv 10 \pmod{12}$, $n \equiv 5 \pmod 6$. Now $K_{m,n} = K_{m,n-4} \cup K_{m,4}$ where the partite sets of $K_{m,n-4}$ are V_m and $V_n \setminus \{1_2, 2_2, 3_2, 4_2\}$ and the partite sets of $K_{m,4}$ are V_m and $\{1_2, 2_2, 3_2, 4_2\}$. There exists an unrestricted hexagon covering of $K_{m,n-4}$ with padding P_1 where $|E(P_1)| = m/2 + 3$ by Case 12 and there is a restricted hexagon covering of $K_{m,4}$ with padding P_2 where $|E(P_2)| = 2$ by Lemma 3.1 Case 2. Therefore there is an unrestricted hexagon covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = m/2 + 5$. ■

Lemma 4.5 *A minimal unrestricted hexagon covering of $K_{m,n}$ where m and n are both odd, $m \geq n \geq 3$, has a padding P satisfying $|E(P)| = (m+n)/2 + k$ where k is the smallest nonnegative integer such that $|E(K_{m,n})| + (m+n)/2 + k \equiv 0 \pmod 6$.*

Proof. Since each vertex of $K_{m,n}$ is of odd degree, in the padding of a covering each of these vertices will be of odd degree. Therefore in an unrestricted covering of $K_{m,n}$ with padding P, it is necessary that $|E(P)| \geq (m+n)/2$. Since a covering yields a decomposition of $K_{m,n} \cup P$, then it is necessary that $|E(K_{m,n})| + |E(P)| \equiv 0 \pmod 6$.
Case 1. Suppose $m \equiv 1 \pmod{12}$, $m \geq 13$, and $n \equiv 7 \pmod{12}$. We have $K_{m,n} = K_{n,n} \cup K_{m-n,n-3} \cup K_{6,3} \cup K_{m-n-6,3}$ where the partite sets of $K_{n,n}$ are $\{1_1, 2_2, \ldots, n_1\}$ and V_n, the partite sets of $K_{m-n,n-3}$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and $\{4_2, 5_2, \ldots, n_2\}$, the partite sets of $K_{6,3}$ are $\{(n+1)_1, (n+2)_1, \ldots, (n+6)_1\}$ and $\{1_2, 2_2, 3_3\}$, and the partite sets of $K_{m-n-6,3}$ are $\{(n+7)_1, (n+8)_1, \ldots, m_1\}$ and $\{1_2, 2_2, 3_2\}$. There is a hexagon decomposition of $K_{n,n} \setminus M$ by Theorem 2.3, where (without loss of generality) $E(M) = \{(i_1, i_2) \mid i = 1, 2, \ldots, n\}$. There is a hexagon decomposition of $K_{m-n,n-3}$ by Theorem 2.2. There is an unrestricted covering of $K_{m-n-6,3}$ with padding P_1 where $|E(P_1)| = (m-n-6)/2$ by Lemma 4.4 Case 1. Taking these decompositions, the covering, and $\{((3i-2)_1, (3i-2)_2, (3i)_1, (3i)_2, (3i-1)_1, (3i-1)_2) \mid i = 1, 2, \ldots, (n-1)/3\} \cup \{((n+1)_1, 1_2, (n+4)_1, (n+5)_1, 2_2, (n+2)_1), (n_1, n_2, (n+6)_1, 3_2, (n+3)_1, (n+2)_1)\}$ yields an unrestricted covering of $K_{m,n}$ with padding P where $E(P) = E(P_1) \cup \{((3i-2)_1, (3i-1)_2), ((3i-1)_1, (3i)_2), ((3i)_1, (3i-2)_2) \mid i = 1, 2, \ldots, (n-1)/3\} \cup \{((n+1)_1, (n+2)_1); ((n+2)_1, (n+3)_1), ((n+4)_1, (n+5)_1), (n_1, (n+2)_1), ((n+6)_1, n_2)\}$ and so $|E(P)| = (m+n)/2 + 1$.
Case 2. First, suppose $m = 3$ and $n \equiv 1 \pmod{12}$, $n \geq 13$. We have $K_{m,n} = K_{3,13} \cup K_{3,n-13}$ where the partite sets of $K_{3,13}$ are $\{1_1, 2_1, 3_1\}$ and $\{1_2, 2_2, \ldots, 13_2\}$, and the partite sets of $K_{3,n-13}$ are $\{1_1, 2_1, 3_1\}$ and $\{14_2, 15_2, \ldots, n_2\}$. Now $K_{3,n-13}$ has an unrestricted covering with padding P_1 where $|E(P_1)| = (n-13)/2$ by Lemma 4.4 Case 1. Next, we note that there is an unrestricted hexagon covering of $K_{3,13}$, namely the set $\{[1_1, 3_2, 3_1, 2_2, 2_1, 1_2]$, $[1_1, 12_2, 10_2, 11_2, 3_1, 13_2]$, $[3_1, 8_2, 10_2, 2_1, 13_2, 5_2]$, $[1_1,$

$6_2, 4_2, 2_1, 7_2, 9_2]\}$ with padding $P_2 = \{((2_1, 1_2), (3_1, 2_2), (1_1, 3_2), (10_2, 11_2), (10_2, 12_2), (5_2, 13_2), (8_2, 10_2), (4_2, 6_2), (7_2, 9_2)\}$ and so $|E(P_2)| = 9$. Therefore there is an unrestricted covering of $K_{m,n}$ with hexagons with padding $P = P_1 \cup P_2$ where $|E(P)| = (m+n)/2 + 1$.

Now suppose $m \equiv 3 \pmod{12}$, $m \geq 15$, and $n \equiv 1 \pmod{12}$, $n \geq 13$. We have $K_{m,n} = K_{7,13} \cup K_{8,13} \cup K_{m-15,13} \cup K_{15,n-13} \cup K_{m-15,n-13}$ where the partite sets of $K_{7,13}$ are $\{1_1, 2_1, \ldots, 7_1\}$ and $\{1_2, 2_2, \ldots, 13_2\}$, the partite sets of $K_{8,13}$ are $\{8_1, 9_1, \ldots, 15_1\}$ and $\{1_2, 2_2, \ldots, 13_2\}$, the partite sets of $K_{m-15,13}$ are $\{16_1, 17_1, \ldots, m_1\}$ and $\{1_2, 2_2, \ldots, 13_2\}$, the partite sets of $K_{m-15,n-13}$ are $\{16_1, 17_1, \ldots, m_1\}$ and $\{14_2, 15_2, \ldots, n_2\}$. Now $K_{7,13}$ has an unrestricted hexagon covering with padding P_1 where $|E(P_1)| = 11$ by Case 1, $K_{8,13}$ has an unrestricted covering with a padding P_2 where $|E(P_2)| = 4$ by Lemma 4.2 Case 9, $K_{m-15,13}$ has an unrestricted covering with a padding P_3 where $|E(P)| = (m-15)/2$, $K_{15,n-13}$ has an unrestricted covering with a padding P_4 where $|E(P_4)| = (n-13)/2$, and there is a hexagon decomposition of $K_{m-15,n-13}$ by Theorem 2.2. Taking these coverings and the decomposition yields an unrestricted covering of $K_{m,n}$ with padding $P = P_1 \cup P_2 \cup P_3 \cup P_4$ where $|E(P)| = (m+n)/2 + 1$.

Case 3. Suppose $m \equiv 5 \pmod{12}$ and $n \equiv 7 \pmod{12}$. We have $K_{m,n} = K_{m-4,n} \cup K_{4,n}$ where that partite sets of $K_{m-4,n}$ are $V_m \setminus \{(m-3)_1, (m-2)_1, (m-1)_1, m_1\}$ and V_n, and the partite sets of $K_{4,n}$ are $\{(m-3)_1, (m-2)_1, (m-1)_1, m_1\}$ and V_n. Now $K_{m-4,n}$ has an unrestricted hexagon covering with padding P_1 where $|E(P_1)| = (m+n-2)/2$ by Case 1, and $K_{4,n}$ has an unrestricted hexagon covering with padding P_2 where $|E(P_2)| = 2$ by Lemma 4.4 Case 5. Taking these two coverings together gives a covering of $K_{m,n}$ with padding P where $|E(P)| = (m+n)/2 + 1$.

Case 4. Suppose $m \equiv 7 \pmod{12}$ and $n \equiv 1 \pmod{12}$, $n \geq 13$. We have $K_{m,n} = K_{m-4,n} \cup K_{4,n}$ where that partite sets of $K_{m-4,n}$ are $V_m \setminus \{(m-3)_1, (m-2)_1, (m-1)_1, m_1\}$ and V_n, and the partite sets of $K_{4,n}$ are $\{(m-3)_1, (m-2)_1, (m-1)_1, m_1\}$ and V_n. Now $K_{m-4,n}$ has an unrestricted hexagon covering with padding P_1 where $|E(P_1)| = (m+n-2)/2$ by Case 2, and $K_{4,n}$ has an unrestricted hexagon covering with padding P_2 where $|E(P_2)| = 2$ by Lemma 4.4 Case 5. Taking these two coverings together gives a covering of $K_{m,n}$ with padding P where $|E(P)| = (m+n)/2 + 1$.

Case 5. Suppose $m \equiv 9 \pmod{12}$ and $n \equiv 7 \pmod{12}$. We have $K_{m,n} = K_{m-8,n} \cup K_{8,n}$ where that partite sets of $K_{m-8,n}$ are $V_m \setminus \{(m-7)_1, (m-6)_1, \ldots, m_1\}$ and V_n, and the partite sets of $K_{8,n}$ are $\{(m-7)_1, (m-6)_1, \ldots, m_1\}$ and V_n. Now $K_{m-8,n}$ has an unrestricted hexagon covering with padding P_1 where $|E(P_1)| = (m+n-6)/2$ by Case 1, and $K_{8,n}$ has an unrestricted hexagon covering with padding P_2 where $|E(P_2)| = 4$ by Lemma 4.4 Case 9. Taking these two coverings together gives a covering of $K_{m,n}$ with padding P where $|E(P)| = (m+n)/2 + 1$.

Case 6. Suppose $m \equiv 11 \pmod{12}$ and $n \equiv 1 \pmod{12}$. We have

$K_{m,n} = K_{m-8,n} \cup K_{8,n}$ where that partite sets of $K_{m-8,n}$ are $V_m \setminus \{(m-7)_1, (m-6)_1, \ldots, m_1\}$ and V_n, and the partite sets of $K_{8,n}$ are $\{(m-7)_1, (m-6)_1, \ldots, m_1\}$ and V_n. Now $K_{m-8,n}$ has an unrestricted hexagon covering with padding P_1 where $|E(P_1)| = (m+n-6)/2$ by Case 2, and $K_{8,n}$ has an unrestricted hexagon covering with padding P_2 where $|E(P_2)| = 4$ by Lemma 4.4 Case 9. Taking these two coverings together gives a covering of $K_{m,n}$ with padding P where $|E(P)| = (m+n)/2+1$.

For the remaining cases, $K_{m,n} = K_{n,n} \cup K_{m-n,n}$ where the partite sets of $K_{n,n}$ are $\{1_1, 2_1, \ldots, n_1\}$ and V_n and the partite sets of $K_{m-n,n}$ are $\{(n+1)_1, (n+2)_1, \ldots, m_1\}$ and V_n. There exists a restricted hexagon covering of $K_{n,n}$ with padding P_1 and an unrestricted hexagon covering of $K_{m-n,n}$ with padding P_2, by previous results. These allow us to cover $K_{m,n}$ with padding $P = P_1 \cup P_2$ which satisfies the required conditions. We present the results in a table which covers these 30 cases.

m (mod 12)	n (mod 12)	$m-n$ (mod 12)	$\|E(P_1)\|$	Lemma/ Case	$\|E(P_2)\|$	Lemma/ Case
1	1	0	$n+4$	3.4/1	$(m-n)/2$	4.4/1
1	3	10	n	3.4/5	$(m-n)/2+1$	4.4/13
1	5	8	n	3.3	$(m-n)/2+4$	4.4/11
1	9	4	n	3.4/5	$(m-n)/2+4$	4.4/6
1	11	2	n	3.3	$(m-n)/2+1$	4.4/4
3	3	0	n	3.4/5	$(m-n)/2$	4.2/1
3	5	10	n	3.3	$(m-n)/2+5$	4.4/14
3	7	8	$n+4$	3.4/1	$(m-n)/2$	4.4/9
3	9	6	n	3.4/5	$(m-n)/2+3$	4.4/8
3	11	4	n	3.3	$(m-n)/2+2$	4.4/7
5	1	4	$n+4$	3.4/1	$(m-n)/2$	4.4/5
5	3	2	n	3.4/5	$(m-n)/2+5$	4.4/3
5	5	0	n	3.3	$(m-n)/2$	4.4/1
5	9	8	n	3.4/5	$(m-n)/2+2$	4.4/10
5	11	6	n	3.3	$(m-n)/2+3$	4.4/8
7	3	4	n	3.4/5	$(m-n)/2+4$	4.4/6
7	5	2	n	3.3	$(m-n)/2+1$	4.4/3
7	7	0	$n+4$	3.4/1	$(m-n)/2$	4.4/1
7	9	10	n	3.4/5	$(m-n)/2+1$	4.4/13
7	11	8	n	3.3	$(m-n)/2+4$	4.4/11
9	1	0	$n+4$	3.4/1	$(m-n)/2$	4.4/9
9	3	10	n	3.4/5	$(m-n)/2+3$	4.4/8
9	5	8	n	3.3	$(m-n)/2+2$	4.4/7
9	9	4	n	3.4/5	$(m-n)/2$	4.4/1
9	11	2	n	3.3	$(m-n)/2+5$	4.4/14
11	3	8	n	3.4/5	$(m-n)/2+1$	4.4/10
11	5	6	n	3.3	$(m-n)/2+4$	4.4/8
11	7	4	$n+4$	3.4/1	$(m-n)/2$	4.4/5
11	9	2	n	3.4/5	$(m-n)/2+4$	4.4/3
11	11	0	n	3.3	$(m-n)/2+1$	4.4/1

■

The results of this section combine to give us the following.

Theorem 4.1 *A minimal unrestricted hexagon covering of $K_{m,n}$ with padding P satisfies:*
(1) *when $m = 1$ and $n \geq 4$, $|E(P)| = 2n$ for n even and $|E(P)| = 2n+3$*

for n odd,
(2) *when* $m = 2$ *and* $n \geq 4$, $|E(P)| = n$ *for n even and* $|E(P)| = n + 3$ *for n odd,*
(3) *when* $m \equiv 0$ *(mod 2),* $m \geq 4$, *and* $n \equiv 1$ *(mod 2),* $n \geq 3$, $|E(P)| = m/2 + k$ *where k is the smallest nonnegative integer such that* $|E(K_{m,n})| + (m/2 + k) \equiv 0$ *(mod 6),*
(4) *when* $m \equiv n \equiv 1$ *(mod 2) and* $m \geq n \geq 3$, $|E(P)| = (m+n)/2+k$ *where k is the smallest nonnegative integer such that* $|E(K_{m,n})|+(m+n)/2+k \equiv 0$ *(mod 6),*
(5) *when* $m \equiv 0$ *(mod 6) and* $n \equiv 0$ *(mod 2),* $n \geq 4$, $|E(L)| = 0$,
(6) *when* $m \equiv n \equiv 2$ *(mod 6),* $n \geq 4$, *or* $m \equiv n \equiv 4$ *(mod 6),* $m \geq 4$, *then* $|E(P)| = 2$, *and*
(7) *when* $m \equiv 2$ *(mod 6),* $m \geq 8$, *and* $n \equiv 4$ *(mod 6), then* $|E(P)| = 4$.

Notice that an unrestricted covering exists when $m = 1$ and when $m = 2$, even though a restricted covering does not exist in these cases. In the cases when (a) $m \equiv 0$ (mod 2), $m \geq 4$, and $n \equiv 1$ (mod 2), $n \geq 3$, and (b) $m \equiv n \equiv 1$ (mod 2) and $m \geq n \geq 3$, both restricted and unrestricted coverings exist, but the unrestricted covering yields a smaller padding by a factor of 2 (roughly).

References

[1] A. Brouwer, Optimal Packings of K_4's into a K_n, *Journal of Combinatorial Theory, Series A* **26**(3) (1979), 278–297.

[2] L. Brown, G. Coker, R. Gardner, and J. Kennedy, Packing the Complete Bipartite Graph with Hexagons, *Congressus Numerantium*, **174** (2005), 97–106.

[3] M. Fort and G. Hedlund, Minimal Coverings of Pairs by Triples, *Pacific Journal of Mathematics* **8** (1958), 709–719.

[4] J. Kennedy, Maximum Packings of K_n with Hexagons, *Australasian Journal of Combinatorics* **7** (1993), 101–110.

[5] J. Kennedy, Maximum Packings of K_n with Hexagons: Corrigendum, *Australasian Journal of Combinatorics* **10** (1994), 293.

[6] J. Kennedy, Minimum Coverings of K_n with Hexagons, *Autstralasian Journal of Combinatorics* **16** (1997), 295–303.

[7] J. Schönheim, On Maximal Systems of k-Tuples, *Studia Sci. Math. Hungarica* (1966), 363–368.

[8] J. Schönheim and A. Bialostocki, Packing and Covering of the Complete Graph with 4-Cycles, *Canadian Mathematics Bulletin* **18**(5) (1975), 703–708.

[9] D. Sotteau, Decompositions of $K_{m,n}$ ($K^*_{m,n}$) into Cycles (Circuits) of Length $2k$, *Journal of Combinatorial Theory, Series B*, **30** (1981), 75-81.