A SPECIAL CASE OF BICYCLIC STEINER TRIPLE SYSTEMS

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Abstract. Necessary and sufficient conditions are given for a Steiner triple system of order v admitting an automorphism consisting of a 3-cycle and a $(v-3)$-cycle.

1. Introduction

A *Steiner triple system* of order v, denoted $STS(v)$ or STS, is a v-element set, X, of points, together with a set β, of unordered triples of elements of X, called *blocks*, such that any two points of X are together in exactly one block of β. It is well known that a $STS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6). An *automorphism* of a $STS(v)$ is a permutation π of X which fixes β. A permutation π of a v-element set is said to be of *type* $[\pi] = [p_1, p_2, ..., p_v]$ if the disjoint cyclic decomposition of π contains p_i cycles of length i. So we have $\sum ip_i = v$. The *orbit* of a block under an automorphism, π, is the image of the block under the powers of π. A set of blocks, B, is said to be a *set of base blocks for a STS(v) under the permutation* π if the orbits of the blocks of B produce the $STS(v)$ and exactly one block of B occurs in each orbit.

A problem of interest has been that of determining the values of v for which there exists a $STS(v)$ admitting an automorphism π of a given type. We denote such a design by $STS_\pi(v)$. An $STS_\pi(v)$ with $[\pi] = [0, 0, \ldots, 0, 1]$ is said to be *cyclic* and such systems exist if and only if $v \equiv 1$ or 3 (mod 6), $v \neq 9$ [4, 6, 8]. If $[\pi] = [1, 0, 0, \ldots, 0, k, 0, \ldots, 0]$ then a $STS_\pi(v)$ is *k-rotational.* Necessary and sufficient conditions exist for k-rotational STSs for $k = 1, 2, 3, 4$, and 6 [2,7]. In an attempt to generalize the idea of a 1-rotational STS, *bicyclic STSs* are introduced in [3]. A bicyclic $STS_\pi(v)$ admits an automorphism of type $[\pi] = [0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 0]$ where $p_N = 1$, $p_M = 1$ and $N + M = v$. For such a $STS_\pi(v)$, the automorphism π^N has N fixed points, and it is rather easy to see that the fixed points of an automorphism of a STS form a subsystem. This means that $N \equiv 1$ or 3 (mod 6), $N \neq 9$ since the N fixed points of π^N induce a cyclic subsystem of the original STS. If $N = 1$, then the $STS_\pi(v)$ is simply 1-rotational. The necessary and sufficient conditions for a bicyclic $STS_\pi(v)$ for $N = 7$ are given in [3] and for $N > 9$ in [1]. The purpose of this paper is to present the necessary and sufficient conditions for a bicyclic $STS(v)$

admitting an automorphism consisting of a 3-cycle and a $(v-3)$-cycle, that is the case $N = 3$.

2. The Constructions

We shall construct the type of $STS_\pi(v)$ mentioned above on the set $X = \{x_0, x_1, x_2\} \bigcup Z_{v-3}$ with the automorphism being $\pi = (x_0, x_1, x_2)(0, 1, \ldots, v-4)$.

We have the following necessary condition:

Lemma 2.1 *If a $STS_\pi(v)$ exists where $[\pi] = [0,0,1,0,\ldots,0,1,0,0,0]$ then $v \equiv 3 \pmod 6$.*

Proof. Such a $STS_\pi(v)$ must contain some block of the form (x_0, y, z) where $y, z \in Z_{v-3}$. By applying π^{v-3} to this block, we see that the block $\left(\pi^{v-3}(x_0), \pi^{v-3}(y), \pi^{v-3}(z)\right) = \left(\pi^{v-3}(x_0), y, z\right)$ must also be a block of the STS. So $\pi^{v-3}(x_0) = x_0$ and $v \equiv 0 \pmod 3$. This combined with $v \equiv 1$ or $3 \pmod 6$ gives $v \equiv 3 \pmod 6$. ∎

The following lemma demonstrates that the necessary conditions of Lemma 2.1 are also sufficient.

Lemma 2.2 *If $v \equiv 3 \pmod 6$ then there exists a $STS_\pi(v)$ where $[\pi] = [0,0,1,0,\ldots,0,1,0,0,0]$.*

Proof. Let $v \equiv 3 \pmod 6$, say $v = 6k + 3$.

case 1. Suppose $k \equiv 2$ or $4 \pmod 6$. Consider the blocks:

$$(x_0, x_1, x_2), (x_0, 0, 3k), (0, 2k, 4k), \left(x_0, \frac{5k}{2}, 5k\right),$$

$$(0, k+r, 2k-1-r) \text{ for } r = 0, 1, \ldots, \frac{k}{2} - 1, \text{ and}$$

$$(0, 2k+1+r, 3k-1-r) \text{ for } r = 0, 1, \ldots, \frac{k}{2} - 2 \text{ (omit if } k = 2).$$

case 2. Suppose $k \equiv 1, 3$ or $5 \pmod 6$. Consider the blocks:

$$(x_0, x_1, x_2), (x_0, 0, 3k), (0, 2k, 4k), \left(x_0, \frac{3k-1}{2}, 3k-1\right),$$

$$(0, k+r, 2k-1-r) \text{ for } r = 0, 1, \ldots, \frac{k-3}{2} \text{ (omit if } k = 1), \text{ and}$$

$$(0, 2k+1+r, 3k-1-r) \text{ for } r = 0, 1, \ldots, \frac{k-3}{2} \text{ (omit if } k = 1).$$

case 3. Suppose $k \equiv 0 \pmod{12}$. Consider the blocks:

$$(x_0, x_1, x_2), (x_0, 0, 3k), (x_0, k-2, 2k-4), (0, 2k, 4k), (0, 2k+1, 3k+1),$$

$$\left(0, \frac{3k}{2}+1, \frac{5k}{2}\right), \left(0, \frac{3k}{2}, 2k-1\right), \left(0, 1, \frac{5k}{4}+1\right),$$
$$\left(0, \frac{5k}{4}+2+r, \frac{7k}{4}-1-r\right) \text{ for } r = 0, 1, \ldots, \frac{k}{4}-3,$$
$$(0, 2k+2+r, 3k-2-r) \text{ for } r = 0, 1, \ldots, \frac{k}{2}-3, \text{ and}$$
$$(0, k+1+r, 2k-2-r) \text{ for } r = 0, 1, \ldots, \frac{k}{4}-2.$$

case 4a. Suppose $k = 6$. Consider the blocks:

(x_0, x_1, x_2), $(x_0, 0, 18)$, $(x_0, 17, 34)$, $(0, 2, 9)$, $(0, 4, 10)$, $(0, 3, 14)$, $(0, 12, 24)$, $(0, 5, 13)$, and $(0, 1, 16)$.

case 4b. Suppose $k \equiv 6 \pmod{12}$, $k > 6$. Consider the blocks:

$$(x_0, x_1, x_2), (x_0, 0, 3k), (x_0, 3k-1, 6k-2), (0, 2k, 4k),$$
$$(0, 2k-2, 4k-3), \left(0, k-7, \frac{3k}{2}-2\right), \left(0, \frac{5k-18}{4}, 3k-4\right),$$
$$\left(0, \frac{3k}{2}-3-r, \frac{3k}{2}-1+r\right) \text{ for } r = 0, 1, \ldots, \frac{k+2}{4}, \frac{k+10}{4}, \frac{k+14}{4}, \ldots, \frac{k}{2}-2,$$
and
$$\left(0, \frac{5k}{2}-1-r, \frac{5k}{2}+r\right) \text{ for } r = 0, 1, \ldots, \frac{k}{2}-5, \frac{k}{2}-3, \frac{k}{2}-2.$$

In each case, the collection of blocks forms a set of base blocks for a $STS_\pi(v)$. ∎

Lemmas 2.1 and 2.2 combine to give us:

Theorem *A $STS_\pi(v)$ where $[\pi] = [0, 0, 1, 0, \ldots, 0, 1, 0, 0, 0]$ exists if and only if $v \equiv 3 \pmod{6}$.*

3. Conclusion

In conclusion, the following results are known concerning bicyclic STSs which admit an automorphism π whose cyclic disjoint decomposition consists of a cycle of length N and a larger cycle of length M:

1. If $N = 1$ and $M = v - 1$ then such a $STS_\pi(v)$ is called 1-rotational and exists if and only if $v \equiv 3$ or $9 \pmod{24}$ [7].

2. If $N = 3$ and $M = v - 3$, we have shown that a $STS_\pi(v)$ exists if and only if $v \equiv 3 \pmod{6}$.

3. If $N = 7$ and $M = v - 7$ then a $STS_\pi(v)$ exists if and only if $v \equiv 21 \pmod{42}$ [3].

4. If $N \equiv 1$ or 3 (mod 6), $N > 9$, and $N \mid M$ then a $STS_\pi(v)$ exists if and only if $v = N + M \equiv 1$ or 3 (mod 6) [1].

These results combine to give a complete classification of bicyclic Steiner triple systems.

REFERENCES

[1] R.S. Calahan and R. Gardner, Bicyclic Steiner triple systems, submitted.

[2] C.J. Cho, Rotational Steiner triple systems, *Discrete Math.* 42 (1982) 153-159.

[3] R. Gardner, A note on a class of Steiner triple systems, *Ars Combinatoria*, to appear.

[4] L. Heffter, Ueber Tripelsysteme, *Math. Ann.* 49 (1897) 101-112.

[5] E.S. O'Keefe, Verification of a conjecture of Th. Skolem, *Math. Scand.* 9 (1961) 80-82.

[6] R. Peltesohn, Eine Lösung der beiden Heffterschen Differenzenprobleme, *Compositio Math.* 6 (1939) 251-257.

[7] K.T. Phelps and A. Rosa, Steiner triple systems with rotational automorphisms, *Discrete Math.* 33 (1981) 57-66.

[8] T. Skolem, On certain distributions of integers in pairs with given differences, *Math. Scand.* 5 (1957) 57-68.