A Note on a Class of Steiner Triple Systems

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Abstract. Steiner triple systems admitting automorphisms whose disjoint cyclic decomposition consist of two cycles are explored. We call such systems *bicyclic*. Several necessary conditions are given. Sufficient conditions are given when the length of the smaller cycle is 7.

1. Introduction

A *Steiner triple system* of order v, denoted STS or $STS(v)$, is a v-element set, X, of points, together with a set β, of unordered triples of elements of X, called *blocks*, such that any two points of X are together in exactly one block of β. It is well known that a $STS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6). An *automorphism* of a STS is a permutation π of X which fixes β. A permutation π of a v-element set is said to be of *type* $[\pi] = [p_1, p_2, \ldots, p_v]$ if the disjoint cyclic decomposition of π contains p_i cycles of length i. So we have $\sum ip_i = v$. The *orbit* of a block under an automorphism, π, is the image of the block under the powers of π. A set of blocks, B, is said to be a *set of base blocks for a STS under the permutation* π if the orbits of the blocks of B produce the STS and exactly one block of B occurs in each orbit.

The question of "For which orders v is there a $STS(v)$ admitting π as an automorphism?" has been explored for several types of automorphisms (for surveys of the results, see [1] and [3]). A *cyclic* $STS(v)$ is one admitting an automorphism of type $[0, 0, \ldots, 1]$ and exists if and only if $v \equiv 1$ or 3 (mod 6) and $v \neq 9$ [4], [5], [6], and [8]. A *k-rotational* STS is one admitting an automorphism of type $[1, 0, \ldots, 0, k, 0, \ldots, 0]$. A 1-rotational $STS(v)$ exists if and only if $v \equiv 3$ or 9 (mod 24), a 2-rotational $STS(v)$ exists if and only if $v \equiv 1, 3, 7, 9, 15$, or 19 (mod 24), and a 6-rotational $STS(v)$ exists if and only if $v \equiv 1, 7$, or 19 (mod 24) [7]. A 3-rotational $STS(v)$ exists if and only if $v \equiv 1$ or 19 (mod 24) and a 4-rotational $STS(v)$ exists if and only if $v \equiv 1$ or 9 (mod 12) [2]. A STS admitting an automorphism of type $[0, 0, 1, 0, \ldots, 0, 1, 0, 0]$ exists if and only if $v \equiv 3$ (mod 6) [1].

Taking our lead from this last result and the 1-rotational result, we are inspired to explore STSs admitting automorphisms consisting of two cycles. There is the obvious temptation, to which we succumb, to call such systems *bicyclic Steiner triple systems*. In this paper, we give several necessary conditions for the existence of these systems along with sufficient conditions when the smaller cycle is of size 7.

2. Necessary conditions for the existence of a bicyclic STS

We first present some necessary conditions for a bicyclic $STS(v)$ admitting an automorphism π of type $[0,0,\ldots,0,p_{N_1},0,\ldots,0,p_{N_2},0,\ldots,0]$ where $p_{N_1} = p_{N_2} = 1$, $N_1 < N_2$ and $N_1 + N_2 = v$.

Lemma 2.1. *A bicyclic $STS(v)$ admitting the above automorphism π satisfies the condition $N_1 \equiv 1$ or $3 (\mathrm{mod}\ 6)$, $N_1 \neq 9$.*

Proof: If we consider the automorphism π^{N_1}, we see that it has N_1 fixed points. As is easily seen, the fixed points of a STS under an automorphism form a subsystem. So $N_1 \equiv 1$ or 3 (mod 6). Also, π restricted to this subsystem shows that the subsystem is itself a cyclic $STS(N_1)$, so $N_1 \neq 9$. ∎

Note that Lemma 2.1 demonstrates that the 1-rotational STS and the STS admitting an automorphism of type $[0,0,1,0,\ldots,0,1,0,0]$ are, in a sense, the first two types of bicyclic Steiner triple systems. The next type would have the smaller cycle of length 7.

Lemma 2.2. *A bicyclic $STS(v)$ admitting the above automorphism π satisfies the condition $N_1 \mid N_2$.*

Proof: Suppose the STS is constructed on the set $X = \mathbf{Z}_{N_1} \times \{1\} \bigcup \mathbf{Z}_{N_2} \times \{2\}$ and that $\pi = (0_1, 1_1, \ldots, (N_1-1)_1)(0_2, 1_2, \ldots, (N_2-1)_2)$. Consider a block of form (a_1, b_2, c_2). Applying π^{N_2} to this block, we get the block $(\pi^{N_2}(a_1), b_2, c_2)$. Hence, $\pi^{N_2}(a_1) = a_1$ and $N_1 \mid N_2$. ∎

Notice that the STS of Lemma 2.2 cannot contain blocks of the form (a_1, b_1, c_2) since applying π^{N_1} to the block, we get the block $(a_1, b_1, \pi^{N_1}(c_2))$, however $\pi^{N_1}(c_2) \neq c_2$ since $N_1 < N_2$.

As in Lemma 2.2, we will consider bicyclic STSs on the set $X = \mathbf{Z}_{N_1} \times \{1\} \bigcup \mathbf{Z}_{N_2} \times \{2\}$, and we use a difference method approach. With the pair (a_i, b_i) we associate the *pure difference of type i* of $\min\{(a_i - b_i)(\mathrm{mod}\ N_i), (b_i - a_i)(\mathrm{mod}\ N_i)\}$. With the pair (a_1, b_2) we associate the *mixed difference* of $(b_2 - a_1)(\mathrm{mod}\ N_1)$.

We are dealing with the difference sets: $\{1, 2, \ldots, (N_1 - 1)/2\}$ of pure differences of type 1, $\{1, 2, \ldots, N_2/2\}$ of pure differences of type 2, and $\{0, 1, \ldots, (N_1 - 1)\}$ of mixed differences.

Lemma 2.3. *A bicyclic $STS(v)$ admitting the above automorphism π satisfies $N_2 \equiv 2$ (mod 6) if $N_1 \equiv 1$ (mod 6).*

Proof: If $N_1 \equiv 1$ (mod 6) then from the condition $v = N_1 + N_2 \equiv 1$ or 3 (mod 6), we get that $N_2 \equiv 0$ or 2 (mod 6). Base blocks of the form (a_1, b_2, c_2) have two mixed differences and one pure difference of type 2 associated with them, with one possible exception. A block of the form $(a_1, 0_2, (N_2/2)_2)$ is an admissible base block (this block has a "short orbit" in the sense that the collection of its images under π is half the size of the set of images of the other admissible blocks of the form (a_1, b_2, c_2)). Since the number of mixed differences, N_1, is odd, we must use one mixed difference in a base block of the form $(a_1, 0_2, (N_2/2))$. The remaining mixed differences are associated in pairs with blocks of the form (a_1, b_2, c_2). Base blocks of the form (a_2, b_2, c_2) have three pure differences of type 2 associated with them, with one possible exception. The block $\beta_1 = (0_2, (N_2/3)_2 (2N_2/3)_2)$ is an admissible base block (this block also has a "short orbit") with the associated difference of $N_2/3$ only. Since each difference must be associated with exactly one base block, we have the conditions:

1. $(N_2/2 - 1) - ((N_1 - 1)/2) \equiv 0 \pmod 3$ if β_1 is not a base block, and
2. $(N_2/2 - 2) - ((N_1 - 1)/2) \equiv 0 \pmod 3$ if β_1 is a base block.

Condition 1 implies that $N_2 \equiv 2 \pmod 6$ and condition 2 implies that $N_2 \equiv 4 \pmod 6$. However, $N_2 \equiv 4 \pmod 6$ is not possible, as shown above. So, $N_2 \equiv 2 \pmod 6$. ∎

Notice from the proof of Lemma 2.3 that if $N_1 \equiv 1 \pmod 6$ then the short orbit block β_1 cannot be a base block.

Lemma 2.4. *A bicyclic $STS(v)$ admitting the above automorphism π satisfies $N_2 \equiv 0 \pmod 6$ if $N_1 \equiv 3 \pmod 6$.*

Proof: If $N_1 \equiv 3 \pmod 6$, $N_1 \neq 9$, then from the condition $v = N_1 + N_2 \equiv 1$ or $3 \pmod 6$ we get that $N_1 \equiv 0$ or $4 \pmod 6$. However, since $N_1 \mid N_2$ is necessary, we see that $N_2 \equiv 4 \pmod 6$ is not possible. ∎

With the counting argument of Lemma 2.3, we see that for $N_1 \equiv 3 \pmod 6$, $N_1 \neq 9$ and $N_2 \equiv 0 \pmod 6$, the short orbit block β_1 *must* be a base block.

3. Sufficient conditions for a bicyclic STS(v) with $N_1 = 7$

We conclude with a theorem.

Theorem 3.1. *A bicyclic $STS(v)$ which admits an automorphism consisting of a 7 cycle and a larger cycle exists if and only if $v \equiv 21 \pmod{42}$.*

Proof: We shall construct such systems on $X = \mathbf{Z}_{N_1} \times \{1\} \bigcup \mathbf{Z}_{N_2} \times \{2\}$. From Lemma 2.2, $7 \mid N_2$ and from Lemma 2.3, $N_2 \equiv 2 \pmod 6$. These two conditions imply $N_2 \equiv 14 \pmod{42}$ and so $v \equiv 21 \pmod{42}$. Suppose $v = 42k + 21$ and $N_2 = 42k + 14$.

First, suppose $k \geq 0$ is even. Then the following are base blocks for a bicyclic $STS(v)$ under the automorphism $\pi = (0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1)(0_2, 1_2, \ldots, (N_2 - 1)_2)$:

$(0_1, 1_1, 3_1), (0_1, 0_2, (21k + 7)_2)$,
$(0_1, 4_2, (21k + 10)_2), (0_1, 1_2, (21k + 6)_2), (0_1, 5_2, (35k/2 + 9)_2)$,
$(0_2, (21k/2 + 2 - r)_2, (21k/2 + 3 + r)_2)$ for $r = 0, 1, \ldots, 7k/2$, and

$(0_2, (35k/2+3-r)_2, (35k/2+5+r)_2)$ for $r = 0, 1, \ldots, 7k/2-1$ (omit if $k = 0$).

Now suppose $k \geq 1$ is odd. Then the following are base blocks for a bicyclic $STS(v)$ under the above mentioned automorphism π:

$(0_1, 1_1, 3_1), (0_1, 0_2, (21k+7)_2)$,
$(0_1, 1_2, (7k+2)_2), (0_1, 5_2, (14k+11)_2), (0_1, 3_2, ((35k+19)/2)_2)$,
$(0_2, ((21k+7)/2-r)_2, ((21k+9)/2+r)_2)$ for $r = 0, 1, \ldots, (7k+1)/2$, and
$(0_2, ((35k+11)/2-r)_2, ((35k+15)/2+r)_2)$ for $r = 0, 1, \ldots, (7k-3)/2$. ∎

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