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Bicyclic Steiner triple systems

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Abstract

A Steiner triple system admitting an automorphism whose disjoint cyclic decomposition consists of two cycles is said to be *bicyclic*. Necessary and sufficient conditions are given for the existence of bicyclic Steiner triple systems.

1. Introduction

A *Steiner triple system* of order v, denoted STS or STS(v), is a v-element set X of points, together with a set β, of unordered triples of elements of X, called *blocks*, such that any two points of X are together in exactly one block of β. It is well known that a STS(v) exists if and only if $v\equiv 1$ or $3 \pmod 6$. An *automorphism* of a STS is a permutation π of X which fixes β. A permutation π of a v-element set is said to be of *type* $[\pi]=[p_1, p_2, \ldots, p_v]$ if the disjoint cyclic decomposition of π contains p_i cycles of length i. The *orbit* of a block under an automorphism, π, is the image of the block under the powers of π. A set of blocks B is said to be a *set of base blocks for a STS* under the permutation π if the orbits of the blocks of B produce the STS and exactly one block of B occurs in each orbit.

Several types of automorphisms have been explored in connection with the question 'for which orders v does there exist a STS(v) admitting an automorphism of the given type?' A *cyclic* STS(v) is one admitting an automorphism of type $[0, 0, \ldots, 1]$ and exists if and only if $v\equiv 1$ or $3 \pmod 6$ and $v\neq 9$ [5, 6, 7, 10]. A *reverse* STS(v) admits an automorphism of type $[1, (v-1)/2, 0, \ldots, 0]$. Reverse STS$(v)$s exist if and only if $v\equiv 1, 3, 9,$ or $19 \pmod{24}$ [3, 9, 11, 12]. A k-rotational STS(v) admits an automorphism of type $[1, 0, 0, \ldots, 0, k, 0, \ldots, 0]$. k-rotational STSs have been addressed for $k=1, 2, 3, 4,$ and 6 [2, 8].

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In this paper, we explore STS(v)s admitting an automorphism of type $[\pi]=[0,0,\dots,0,p_{N_1},0,\dots,0,p_{N_2},0,\dots,0]$ where $p_{N_1}=p_{N_2}=1$, $N_1<N_2$ and $N_1+N_2=v$. That is, the disjoint cyclic decomposition of π consists of one cycle of length N_1 and another (larger) cycle of length N_2. We call such systems *bicyclic Steiner triple systems.*

2. Previous results and necessary conditions

A bicyclic STS(v) with the smaller cycle of length 1 is also a 1-rotational STS(v) and exists if and only if $v\equiv 3$ or $9 \pmod{24}$ [8]. So, henceforth, we will assume $N_1>1$. If $N_1=3$, then a bicyclic STS(v) admitting an automorphism of type $[0,0,1,0,\dots,0,1,0,0,0]$ exists if and only if $v\equiv 3 \pmod 6$ [1]. If $N_1=7$, then a bicyclic STS(v) admitting the relevant type of automorphism exists if and only if $v\equiv 21 \pmod{42}$ [4]. Some necessary conditions, in particular the following lemmas, for the existence of bicyclic STS(v)s were given in [4].

Lemma 2.1. *A bicyclic* STS(v) *admitting the above automorphism* π *satisfies the condition* $N_1\equiv 1$ *or* $3 \pmod 6$, $N_1\neq 9$, *and* $N_1 \mid N_2$.

Lemma 2.2. *A bicyclic* STS(v) *admitting the above automorphism* π *with* $N_1\equiv 1 \pmod 6$ *satisfies the condition,* $N_2\equiv 2 \pmod 6$. *If* $N_1\equiv 3 \pmod 6$ *then* $N_2\equiv 0 \pmod 6$ *is necessary.*

We will show that these necessary conditions are sufficient for $N_1>1$.

In our constructions, we will require the use of two structures. An (A,n)*-system* is a collection of ordered pairs (a_r,b_r) for $r=1,2,\dots,n$ that partition the set $\{1,2,\dots,2n\}$ with the property that $b_r=a_r+r$ for $r=1,2,\dots,n$. An (A,n)-system exists if and only if $n\equiv 0$ or $1 \pmod 4$ [10]. A (B,n)*-system* is a collection of ordered pairs (a_r,b_r) for $r=1,2,\dots,n$ that partition the set $\{1,2,\dots,2n-1,2n+1\}$ with the property that $b_r=a_r+r$ for $r=1,2,\dots,n$. These systems exist if and only if $n\equiv 2$ or $3 \pmod 4$ [6].

3. Constructions and sufficient conditions

In this section, we present several constructions to show that the necessary conditions of Lemmas 2.1 and 2.2 are sufficient. We will construct bicyclic STS(v)s on the set $X=\mathbf{Z}_{N_1}\times\{1\}\cup Z_{N_2}\times\{2\}$ with the automorphism being $\pi=(0_1,1_1,\dots,(N_1-1)_1)(0_2,1_2,\dots,(N_2-1)_2)$. Since $N_1 \mid N_2$, we will let $N_2=kN_1$ and express base blocks in terms of k and N_1.

Lemma 3.1. *A bicyclic* STS(v) *on the set* X *admitting the automorphism* π *exists if:*

$N_1 \equiv 1 \pmod{24}$ *and* $k \equiv 2 \pmod{24}$, *or*

$N_1 \equiv 1 \pmod{24}$ *and* $k \equiv 8 \pmod{24}$, *or*

$N_1 \equiv 13 \pmod{24}$ *and* $k \equiv 14 \pmod{24}$, *or*

$N_1 \equiv 13 \pmod{24}$ *and* $k \equiv 20 \pmod{24}$.

Proof. Under any one of these conditions,

$$3M = \frac{N_2}{2} - \frac{N_1 - 1}{2} - 1 \equiv 0 \text{ or } 3 \pmod{12}$$

and $M \equiv 0$ or $1 \pmod 4$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{N_1-1}{4}+r\right)_2, \left(\frac{(2k+1)N_1-5}{4}-r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-5}{4},$$

$$\left(0_1, \left(\frac{3N_1+1}{4}+r\right)_2, \left(\frac{(2k+3)N_1-7}{4}-r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-5}{4},$$

$$\left(0_1, \left(\frac{3N_1-3}{4}\right)_2, \left(\frac{(2k+3)N_1-3}{4}\right)_2\right),$$

and $(0_2, r_2, (b_r+M)_2)$ for $r=1, 2, \ldots, M$, where the a_r and b_r are from an (A,M)-system.

This collection of blocks along with the base blocks for a cyclic STS(N_1) on $\mathbf{Z}_{N_1} \times \{1\}$ under the automorphism $(0_1, 1_1, \ldots, (N_1-1)_1)$ form a complete set of base blocks for a bicyclic STS(v) with $v = N_1 + N_2$. □

Lemma 3.2. *A bicyclic* STS(v) *on the set* X *admitting the automorphism* π *exists if*:

$N_1 \equiv 7 \pmod{24}$ *and* $k \equiv 2 \pmod{24}$, *or*

$N_1 \equiv 7 \pmod{24}$ *and* $k \equiv 8 \pmod{24}$, *or*

$N_1 \equiv 19 \pmod{24}$ *and* $k \equiv 14 \pmod{24}$, *or*

$N_1 \equiv 19 \pmod{24}$ *and* $k \equiv 20 \pmod{24}$.

Proof. Under any one of these conditions,

$$3M = \frac{N_2}{2} - \frac{N_1 - 1}{2} - 1 \equiv 0 \text{ or } 3 \pmod{12}$$

and $M \equiv 0$ or 1 (mod 4). Consider the following collection of blocks:

$$\left(0_1,\left(\frac{N_1+1}{4}+r\right)_2,\left(\frac{(2k+1)N_1-7}{4}-r\right)_2\right)\quad \text{for } r=0,1,\ldots,\frac{N_1-7}{4},$$

$$\left(0_1,\left(\frac{3N_1-1}{4}+r\right)_2,\left(\frac{(2k+3)N_1-5}{4}-r\right)_2\right)\quad \text{for } r=0,1,\ldots,\frac{N_1-3}{4},$$

$$\left(0_1,\left(\frac{N_1-3}{4}\right)_2,\left(\frac{(2k+1)N_1-3}{4}\right)_2\right),$$

and $(0_2,r_2,(b_r+M)_2)$ for $r=1,2,\ldots,M$, where the a_r and b_r are from an (A,M)-system.

This collection of blocks along with the base blocks for a cyclic STS(N_1) on $\mathbf{Z}_{N_1}\times\{1\}$ under the automorphism $(0_1,1_1,\ldots,(N_1-1)_1)$ form a complete set of base blocks for a bicyclic STS(v) with $v=N_1+N_2$. □

Lemma 3.3. *A bicyclic* STS(v) *on the set X admitting the automorphism* π *exists if*:

$N_1\equiv 1$ (mod 24) *and* $k\equiv 14$ (mod 24), *or*

$N_1\equiv 1$ (mod 24) *and* $k\equiv 20$ (mod 24), *or*

$N_1\equiv 13$ (mod 24) *and* $k\equiv 2$ (mod 24), *or*

$N_1\equiv 13$ (mod 24) *and* $k\equiv 8$ (mod 24).

Proof. Under any one of these conditions,

$$3M=\frac{N_2}{2}-\frac{N_1-1}{2}-1\equiv 0 \text{ or } 3 \pmod{12}$$

and $M\equiv 2$ or 3 (mod 4). Consider the following collection of blocks:

$$\left(0_1,\left(\frac{N_1-1}{4}+r\right)_2,\left(\frac{(2k+1)N_1-5}{4}-r\right)_2\right)\quad \text{for } r=0,1,\ldots,\frac{N_1-5}{4},$$

$$\left(0_1,\left(\frac{3N_1+1}{4}+r\right)_2,\left(\frac{(2k+3)N_1-7}{4}-r\right)_2\right)\quad \text{for } r=0,1,\ldots,\frac{N_1-9}{4},$$

$$\left(0_1,\left(\frac{N_1-1}{2}\right)_2,\left(\frac{kN_1}{2}-1\right)_2\right),\left(0_1,\left(\frac{3N_1-3}{4}\right)_2,\left(\frac{(2k+3)-3}{4}\right)_2\right)$$

and $(0_2,r_2,(b_r+M)_2)$ for $r\equiv 1,2,\ldots,M$, where the a_r and b_r are from a (B,M)-system.

This collection of blocks along with the base blocks for a cyclic STS(N_1) on $\mathbf{Z}_{N_1}\times\{1\}$ under the automorphism $(0_1,1_1,\ldots,(N_1-1)_1)$ form a complete set of base blocks for a bicyclic STS(v) with $v=N_1+N_2$. □

Lemma 3.4. *A bicyclic* STS(v) *on the set* X *admitting the automorphism* π *exists if*:

$N_1 \equiv 7 \pmod{24}$ *and* $k \equiv 14 \pmod{24}$, *or*

$N_1 \equiv 7 \pmod{24}$ *and* $k \equiv 20 \pmod{24}$, *or*

$N_1 \equiv 19 \pmod{24}$ *and* $k \equiv 2 \pmod{24}$, *or*

$N_1 \equiv 19 \pmod{24}$ *and* $k \equiv 8 \pmod{24}$.

Proof. Under any one of these conditions,

$$3M = \frac{N_2}{2} - \frac{N_1 - 1}{2} - 1 \equiv 0 \text{ or } 3 \pmod{12}$$

and $M \equiv 2$ or $3 \pmod 4$. Consider the following collection of blocks:

$$\left(0_1, \left(\frac{N_1+1}{4}+r\right)_2, \left(\frac{(2k+1)N_1-7}{4}-r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-7}{4},$$

$$\left(0_1, \left(\frac{3N_1-1}{4}+r\right)_2, \left(\frac{(2k+3)N_1-5}{4}-r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-7}{4},$$

$$\left(0_1, \left(\frac{N_1-3}{4}\right)_2, \left(\frac{(2k+1)N_1-3}{4}\right)_2\right), \left(0_1, \left(\frac{N_1-1}{2}\right)_2, \left(\frac{kN_1}{2}-1\right)_2\right)$$

and $(0_2, r_2, (b_r+M)_2)$ for $r=1,2,\ldots,M$, where the a_r and b_r are from a (B,M)-system.
This collection of blocks along with the base blocks for a cyclic STS(N_1) on $\mathbf{Z}_{N_1} \times \{1\}$ under the automorphism $(0_1, 1_1, \ldots, (N_1-1)_1)$ form a complete set of base blocks for a bicyclic STS(v) with $v = N_1 + N_2$. $\square$

Lemmas 3.1–3.4 combine to give us the following theorem.

Theorem 3.5. *A bicyclic* STS(v) *where* $v = N_1 + N_2$ *and* $N_2 = kN_1$ *admitting an automorphism whose disjoint cyclic decomposition is a cycle of length* N_1 *and a cycle of length* N_2 *exists if* $N_1 \equiv 1 \pmod 6$, $N_1 > 1$, *and* $k \equiv 2 \pmod 6$.

We now turn our attention to the case when $N_1 \equiv 3 \pmod 6$.

Lemma 3.6. *A bicyclic* STS(v) *on the set* X *admitting the automorphism* π *exists if* $N_1 \equiv 3 \pmod{12}$, $N_1 > 3$, *and* $k \equiv 0 \pmod{12}$.

Proof. Consider the following collection of base blocks:

$$\left(0_2, \left(\frac{kN_1-24}{6}-2r\right)_2, \left(\frac{kN_1-3}{3}-r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{kN_1-36}{12},$$

$$\left(0_2,\left(\frac{kN_1-18}{6}-2r\right)_2,\left(\frac{kN_1-2}{2}-r\right)_2\right)$$

for $r=\frac{N_1-9}{6},\frac{N_1-3}{6},\ldots,\frac{kN_1-24}{12}$,

$$\left(0_2,\left(\frac{kN_1+6}{6}\right)_2,\left(\frac{kN_1+3}{3}\right)_2\right),\left(0_2,\left(\frac{kN_1}{3}\right)_2,\left(\frac{2kN_1}{3}\right)_2\right),$$

$$\left(0_1,\left(\frac{N_1-3}{6}+r\right)_2,\left(\frac{(k+1)N_1-21}{6}-r\right)_2\right)$$

for $r=0,1,\ldots,\frac{N_1-21}{6}$ (omit if $N_1=15$),

$$\left(0_1,\left(\frac{5N_1-63}{12}-r\right)_2,\left(\frac{(4k+5)N_1-39}{12}+r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-27}{12}$$

(omit if $N_1=15$),

$$\left(0_1,\left(\frac{7N_1-57}{12}+r\right)_2,\left(\frac{(6k+7)N_1-69}{12}-r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-15}{12},$$

$$\left(0_1,\left(\frac{3N_1-29}{4}-r\right)_2,\left(\frac{(4k+9)N_1-51}{12}+r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-27}{12}$$

(omit if $N_1=15$),

$$\left(0_1,\left(\frac{11N_1-57}{12}+r\right)_2,\left(\frac{(6k+11)N_1-81}{12}-r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-27}{2}$$

(omit if $N_1=15$),

$$\left(0_1,\left(\frac{N_1-15}{6}\right)_2,\left(\frac{(3k+2)N_1-18}{12}\right)_2\right),\left(0_1,\left(\frac{3N_1-21}{4}\right)_2,\right.$$

$$\left.\left(\frac{(4k+11)N_1-69}{12}\right)_2\right),\left(0_1,(N_1-5)_2,\left(\frac{(k+6)N_1-36}{6}\right)_2\right),$$

$$\left(0_1,\left(\frac{3N_1-25}{4}\right)_2,\left(\frac{(2k+5)N_1-51}{12}\right)_2\right),$$

$$\left(0_1,(N_1-1)_2,\left(\frac{(k+6)N_1-18}{6}\right)_2\right),\left(0_1,(N_1-4)_2,\left(\frac{(k+6)N_1-12}{6}\right)_2\right),$$

and $\left(0_1\left(\frac{5N_1-33}{6}\right)_2,\left(\frac{(3k+5)N_1-33}{6}\right)_2\right).$

This collection of blocks along with the base blocks for a cyclic STS(N_1) on $\mathbf{Z}_{N_1} \times \{1\}$ under the automorphism $(0_1, 1_1, \ldots, (N_1-1)_1)$ form a complete set of base blocks for a bicyclic STS(v) with $v = N_1 + N_2$. □

Lemma 3.7. *A bicyclic* STS(v) *on the set* X *admitting the automorphism* π *exists if* $N_1 \equiv 3 \pmod{12}$, $N_1 > 3$, *and* $k \equiv 6 \pmod{12}$.

Proof. Consider the following collection of base blocks:

$$\left(0_2, \left(\frac{kN_1-18}{6}-2r\right)_2, \left(\frac{kN_1-2}{2}-r\right)_2\right)$$

for $r = \frac{N_1-9}{6}, \frac{N_1-3}{6}, \ldots, \frac{kN_1-30}{12}$,

$$\left(0_2, \left(\frac{kN_1-24}{6}-2r\right)_2, \left(\frac{kN_1-3}{3}-r\right)_2\right) \quad \text{for } r = 0, 1, \ldots, \frac{kN_1-30}{12},$$

$$\left(0_2, \left(\frac{kN_1}{6}\right)_2, \left(\frac{kN_1+3}{3}\right)_2\right), \left(0_2, \left(\frac{kN_1}{3}\right)_2, \left(\frac{2kN_1}{3}\right)_2\right),$$

$$\left(0_1, \left(\frac{N_1-9}{6}+r\right)_2, \left(\frac{(k+1)N_1-15}{6}-r\right)_2\right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1-15}{6},$$

$$\left(0_1, \left(\frac{5N_1-51}{12}-r\right)_2, \left(\frac{(4k+5)N_1-27}{12}+r\right)_2\right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1-15}{12},$$

$$\left(0_1, \left(\frac{7N_1-21}{12}+r\right)_2, \left(\frac{(6k+7)N_1-45}{12}-r\right)_2\right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1-27}{12}$$

(omit if $N_1 = 15$),

$$\left(0_1, \left(\frac{3N_1-13}{4}+r\right)_2, \left(\frac{(2k+3)N_1-17}{4}-r\right)_2\right) \quad \text{for } r = 0, 1, \ldots, \frac{N_1-15}{12},$$

$$\left(0_1, \left(\frac{11N_1-69}{12}-r\right)_2, \left(\frac{(4k+11)kN_1-33}{12}+r\right)_2\right)$$

for $r = 0, 1, \ldots, \frac{N_1-27}{12}$ (omit if $N_1 = 15$),

$$\left(0_1, (N_1-1)_2, \left(\frac{(k+6)N_1-18}{6}\right)_2\right), \left(0_1, (N_1-4)_2, \left(\frac{(k+6)N_1-12}{6}\right)_2\right),$$

$$\left(0_1, \left(\frac{11N_1-45}{12}\right)_2, \left(\frac{(5k+11)N_1-39}{12}\right)_2\right), \left(0_1, \left(\frac{11N_1-57}{12}\right)_2,\right.$$

$$\left.\left(\frac{(2k+7)N_1-33}{12}\right)_2\right), \text{ and } \left(0_1\left(\frac{N_1-5}{2}\right)_2, \left(\frac{(k+1)N_1-5}{2}\right)_2\right).$$

This collection of blocks along with the base blocks for a cyclic STS(N_1) on $\mathbf{Z}_{N_1}\times\{1\}$ under the automorphism $(0_1, 1_1, \ldots, (N_1-1)_1)$ form a complete set of base blocks for a bicyclic STS(v) with $v=N_1+N_2$. □

Lemma 3.8. *A bicyclic* STS(v) *on the set X admitting the automorphism* π *exists if* $N_1\equiv 9 \pmod{12}$, $N_1>9$, *and* $k\equiv 0 \pmod{12}$.

Proof. Consider the following collection of base blocks:

$$\left(0_2, \left(\frac{kN_1+18}{6}+r\right)_2, \left(\frac{kN_1-3}{3}-r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{kN_1-36}{12},$$

$$\left(0_2, \left(\frac{kN_1+6}{3}+r\right)_2, \left(\frac{kN_1-2}{2}-r\right)_2\right)$$

$$\text{for } r=\frac{N_1-9}{6}, \frac{N_1-3}{6}, \ldots, \frac{N_1-24}{12},$$

$$\left(0_2, (N_1+1)_2, \left(\frac{kN_1+3}{3}\right)_2\right), \left(0_2, \left(\frac{kN_1}{3}\right)_2, \left(\frac{2kN_1}{3}\right)_2\right),$$

$$\left(0_1, \left(\frac{N_1-3}{6}+r\right)_2, \left(\frac{(k+1)N_1-21}{6}-r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-21}{6},$$

$$\left(0_1, \left(\frac{5N_1-57}{12}-r\right)_2, \left(\frac{(4k+5)N_1-33}{12}+r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-21}{12},$$

$$\left(0_1, \left(\frac{2N_1-18}{3}-r\right)_2, \left(\frac{(k+1)N_1-7}{2}+r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-21}{12},$$

$$\left(0_1, \left(\frac{3N_1-19}{4}+r\right)_2, \left(\frac{(2k+3)N_1-27}{4}-r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-21}{12},$$

$$\left(0_1, \left(\frac{11N_1-87}{12}-r\right)_2, \left(\frac{(4k+11)N_1-51}{12}+r\right)_2\right) \quad \text{for } r=0,1,\ldots,\frac{N_1-21}{12},$$

$$\left(0_1, \left(\frac{3N_1-23}{4}\right)_2, \left(\frac{(2k+5)N_1-45}{12}\right)_2\right), \left(0_1, \left(\frac{N_1-15}{6}\right)_2, \left(\frac{(3k+2)N_1-18}{12}\right)_2\right),$$

$$\left(0_1, \left(\frac{11N_1-63}{12}\right)_2, \left(\frac{(2k+11)N_1-75}{12}\right)_2\right), \left(0_1, (N_1-1)_2, \left(\frac{(k+6)N_1-18}{6}\right)_2\right),$$

$$\left(0_1, (N_1-4)_2, \left(\frac{(k+6)N_1-12}{6}\right)_2\right), \text{ and } \left(0_1, (N_1-5)_2, \left(\frac{(k+2)N_1-10}{2}\right)_2\right).$$

This collection of blocks along with the base blocks for a cyclic STS(N_1) on $\mathbf{Z}_{N_1} \times \{1\}$ under the automorphism $(0_1, 1_1, \ldots, (N_1-1)_1)$ form a complete set of base blocks for a bicyclic STS(v) with $v = N_1 + N_2$. □

Lemma 3.9. *A bicyclic* STS(v) *on the set* X *admitting the automorphism* π *exists if* $N_1 \equiv 9 \pmod{12}$, $N_1 > 9$, *and* $k \equiv 6 \pmod{12}$.

Proof. Consider the following collection of base blocks:

$$\left(0_2, \left(\frac{kN_1-18}{6}-2r\right)_2, \left(\frac{kN_1-2}{2}-r\right)_2\right)$$

for $r = \frac{N_1-9}{6}, \frac{N_1-3}{6}, \ldots, \frac{kN_1-30}{12}$,

$$\left(0_2, \left(\frac{kN_1-24}{6}-2r\right)_2, \left(\frac{kN_1-3}{3}-r\right)_2\right) \quad \text{for } r=0, 1, \ldots, \frac{kN_1-30}{12},$$

$$\left(0_2, \left(\frac{kN_1}{6}\right)_2, \left(\frac{kN_1+3}{3}\right)_2\right), \left(0_2, \left(\frac{kN_1}{3}\right)_2, \left(\frac{2kN_1}{3}\right)_2\right),$$

$$\left(0_1, \left(\frac{N_1-3}{6}+r\right)_2, \left(\frac{(k+1)N_1-9}{6}-r\right)_2\right) \quad \text{for } r=0, 1, \ldots, \frac{N_1-9}{6},$$

$$\left(0_1, \left(\frac{5N_1-45}{12}-r\right)_2, \left(\frac{(4k+5)N_1-21}{12}+r\right)_2\right) \quad \text{for } r=0, 1, \ldots, \frac{N_1-33}{12}$$

(omit if $N_1 = 21$),

$$\left(0_1, \left(\frac{7N_1-63}{12}-r\right)_2, \left(\frac{(4k+7)N_1-27}{12}+r\right)_2\right) \quad \text{for } r=0, 1, \ldots, \frac{N_1-21}{12},$$

$$\left(0_1, \left(\frac{3N_1-7}{4}+r\right)_2, \left(\frac{(2k+3)N_1-15}{4}-r\right)_2\right) \quad \text{for } r=0, 1, \ldots, \frac{N_1-21}{12},$$

$$\left(0_1, \left(\frac{11N_1-27}{12}+r\right)_2, \left(\frac{(6k+11)N_1-63}{12}-r\right)_2\right) \quad \text{for } r=0, 1, \ldots, \frac{N_1-33}{12}$$

(omit if $N_1 = 21$),

$$\left(0_1, \left(\frac{7N_1-39}{12}\right)_2, \left(\frac{(6k+7)N_1-51}{12}\right)_2\right), \left(0_1, \left(\frac{3N_1-11}{4}\right)_2, \left(\frac{(4k+11)N_1-51}{12}\right)_2\right),$$

$$\left(0_1, \left(\frac{11N_1-39}{12}\right)_2, \left(\frac{(5k+11)N_1-33}{12}\right)_2\right), \left(0_1, (N_1-1)_2, \left(\frac{(k+6)N_1-18}{6}\right)_2\right),$$

$$\left(0_1, (N_1-4)_2, \left(\frac{(k+6)N_1-12}{6}\right)_2\right), \text{ and } \left(0_1, \left(\frac{2N_1-9}{3}\right)_2, \left(\frac{(3k+4)N_1-18}{6}\right)_2\right).$$

This collection of blocks along with the base blocks for a cyclic STS(N_1) on $\mathbf{Z}_{N_1} \times \{1\}$ under the automorphism $(0_1, 1_1, \dots, (N_1-1)_1)$ form a complete set of base blocks for a bicyclic STS(v) with $v = N_1 + N_2$. □

Lemmas 3.6–3.9 combine to give us the following theorem.

Theorem 3.10. *A bicyclic* STS(v) *where* $v = N_1 + N_2$ *and* $N_2 = kN_1$ *admitting an automorphism whose disjoint cyclic decomposition is a cycle of length* N_1 *and a cycle of length* N_2 *exists if* $N_1 \equiv 3 \pmod 6$, $N_1 > 9$, *and* $k \equiv 0 \pmod 6$.

Combining Theorems 3.5 and 3.10 with the previous mentioned results for $N_1 = 3$ and $N_1 = 7$ gives us the necessary and sufficient conditions.

Theorem 3.11. *A bicyclic* STS(v) *where* $v = N_1 + N_2$ *admitting an automorphism whose disjoint cyclic decomposition is a cycle of length* N_1, *where* $N_1 > 1$, *and a cycle of length* N_2 *exists if and only if* $N_1 \equiv 1$ *or* $3 \pmod 6$, $N_1 \neq 9$, $N_1 \mid N_2$, *and* $v = N_1 + N_2 \equiv 1$ *or* $3 \pmod 6$.

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