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Steiner triple systems with transrotational automorphisms

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Abstract

A Steiner triple system of order v is said to be k-transrotational if it admits an automorphism consisting of a fixed point, a transposition, and k cycles of length $(v-3)/k$. Necessary and sufficient conditions are given for the existence of 1- and 2-transrotational Steiner triple systems.

1. Introduction

A *Steiner triple system* or order v, denoted $STS(v)$, is a v-element set X of points, together with a set β, of unordered triples of elements of X, called *blocks*, such that any two points of X are together in exactly one block of β. It is well known that a $STS(v)$ exists if and only if $v\equiv 1$ or $3 \pmod 6$. An *automorphism* of a $STS(v)$ is a permutation π of X which fixes β. A permutation π of a v-element set is said to be of *type* $[\pi]=[p_1, p_2, \ldots, p_v]$ if the disjoint cyclic decomposition of π contains p_i cycles of length i. So we have $\sum ip_i = v$. The *orbit* of a block under an automorphism π is the image of the block under the powers of π. A set of blocks B is said to be a *set of base blocks for a $STS(v)$ under the permutation π* if the orbits of the blocks of B produce the $STS(v)$ and exactly one block of B occurs in each orbit.

A question of interest is the following: Given a permutation of type $[\pi]$, for what orders v does there exist a $STS(v)$ with π as an automorphism? This question has been addressed for several particular types of automorphisms:

(i) If $[\pi]=[0,0,\ldots,1]$ then an admissible $STS(v)$ is said to be *cyclic*. A cyclic $STS(v)$ exists if and only if $v\equiv 1$ or $3 \pmod 6$ and $v\neq 9$ (see [3, 7–9, 11, 13]).

(ii) If $[\pi]=[1, \frac{1}{2}(v-1), 0, \ldots, 0]$ then an admissible $STS(v)$ is called *reverse*. A reverse $STS(v)$ exists if and only if $v\equiv 1, 3, 9,$ or $19 \pmod{24}$ (see [4, 5, 12, 14, 15]).

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(iii) If $[\pi]=[1,0,\ldots,0,k,0,\ldots,0]$ then an admissible $STS(v)$ is said to be *k-rotational*. Phelps and Rosa [10] dealt with the cases of $k=1,2$, and 6. A 1-rotational $STS(v)$ exists if and only if $v\equiv 3$ or $9 \pmod{24}$. A 2-rotational $STS(v)$ exists if and only if $v\equiv 1,3,7,9,15$, or $19 \pmod{24}$. A 6-rotational $STS(v)$ exists if and only if $v\equiv 1,7$, or $19 \pmod{24}$. Cho [2] addressed the cases of $k=3$ and $k=4$. A 3-rotational $STS(v)$ exists if and only if $v\equiv 1$ or $19 \pmod{24}$. A 4-rotational $STS(v)$ exists if and only if $v\equiv 1$ or $9 \pmod{12}$.

(iv) If $[\pi]=[f,\frac{1}{2}(v-f),0,\ldots,0]$ where $f>1$, then there exists a $STS(v)$ admitting π as an automorphism if and only if $v\equiv 1$ or $3 \pmod 6$, $f\equiv 1$ or $3 \pmod 6$ and $(v-f\equiv 0 \pmod 4$ and $v\geqslant 2f+1)$ or $(v-f\equiv 2 \pmod 4$ and $v\geqslant 3f)$ [6].

(v) If $[\pi_c]=[1,0,0,t,0,\ldots,0,1,0,\ldots,0]$ then there exists a $STS(v)$ admitting π_c as an automorphism if and only if $v\equiv 9 \pmod{24}$, $t\equiv 2 \pmod 3$, and $t\leqslant\lceil (3e-4)/4\rceil$ with strict inequality if $e\equiv 6,12,14,18$, or $20 \pmod{24}$ where $e=(v-(4t+1))/4$. This result is due to Callahan [1].

We are concerned with automorphisms of the type $[\pi_k]=[1,1,0,\ldots,0,k,0,\ldots,0]$. A $STS(v)$ admitting π_k as an automorphism will be called *k-transrotational*. In this paper we address the problem of necessary and sufficient conditions for the existence of k-transrotational $STS(v)$ for $k=1$ and 2.

2. The existence of 1-transrotational Steiner triple systems

A 1-transrotational $STS(v)$ admits an automorphism of the type $[1,1,0,\ldots,0,1,0,0,0]$. We will construct these on the set $\{\infty,a,b\}\cup Z_N$ where the automorphism is $\pi_1=(\infty)(a,b)(0,1,\ldots,N-1)$.

Lemma 2.1. *A necessary condition for the existence of a* 1*-transrotational* $STS(v)$ *is that* $v\equiv 1,7,9$, *or* $15 \pmod{24}$.

Proof. Since $v\equiv 1$ or $3 \pmod 6$, we have $N\equiv 0$ or $4 \pmod 6$ and $N\equiv 0,4,6,10,12,16,18$, or $22 \pmod{24}$. One of the blocks of the desired $STS(v)$ will be $(\infty,0,\frac{1}{2}N)$. Another will be $(a,0,x)$ where $x\equiv 1 \pmod 2$. So now, the set $\{0,1,\ldots,\frac{1}{2}N\}-\{x,\frac{1}{2}N\}$ must be partitioned into *difference triples* (i.e. triples of distinct positive integers where either two of the numbers sum to the third, or the sum of all three is N). So in this set, there must be an even number of odd numbers since each difference triple includes either 0 or 2 odd numbers. From this condition, it follows that $v\equiv 1,7,9$, or $15 \pmod{24}$. □

Theorem 2.2. *A* 1*-transrotational* $STS(v)$ *exists if and only if* $v\equiv 1,7,9$, or $15 \pmod{24}$.

Proof. Lemma 2.1 gives the necessary conditions. If $v\equiv 1,7,9$ or $15 \pmod{24}$ then $N\equiv 4,6,12$, or $22 \pmod{24}$. We show sufficiency in four cases. In each case, add the blocks $(\infty,0,\frac{1}{2}N)$, $(a,0,x)$, and (∞,a,b).

Case 1: If $N=4$ then let $x=1$. If $N\equiv 4 \pmod{24}$, $N>4$ say $N=24s+4$, $s\geqslant 1$ then take the blocks

$$(0,4s+1+r,8s-r) \quad \text{for } r=0,1,\dots,2s-1,$$

$$(0,8s+1+r,12s+1-r) \quad \text{for } r=0,1,\dots,2s-1,$$

and let $x=10s+1$.

Case 2: If $N=6$ then let $x=1$ and take the block $(0,2,4)$. If $N\equiv 6 \pmod{24}$, $N>6$, say $N=24s+6$, $s\geqslant 1$ then take the blocks

$$(0,4s+1+r,8s+1-r) \quad \text{for } r=0,1,\dots,2s-1,$$

$$(0,4s-1-2r,12s+2-r) \quad \text{for } r=0,1,\dots,2s-1,$$

$$(0,8s+2,16s+4), \text{ and let } x=6s+1.$$

Case 3: If $N=12$ then let $x=5$ and take the blocks $(0,1,3)$ and $(0,4,8)$. If $N\equiv 12 \pmod{24}$, $N>12$ say $N=24s+12$, $s\geqslant 1$, then take the blocks

$$(0,4s+2+r,8s+3-r) \quad \text{for } r=0,1,\dots,2s,$$

$$(0,4s-2r,12s+5-r) \quad \text{for } r=0,1,\dots,2s-1,$$

$$(0,8s+4,16s+8), \text{ and let } x=10s+5.$$

Case 4: If $N=22$ then let $x=9$ and take the blocks $(0,5,6)$ $(0,8,10)$, and $(0,4,7)$. If $N\equiv 22 \pmod{24}$, $N>22$, say $N=24s+22$, $s\geqslant 1$, then take the blocks

$$(0,4s+4+r,8s+7-r) \quad \text{for } r=0,1,\dots,2s+1,$$

$$(0,4s+2-2r,12s+10-r) \quad \text{for } r=0,1,\dots,2s, \text{ and let } x=10s+9.$$

In each case, the blocks are the base blocks for a $STS(v)$ under the automorphism π_1. □

3. The existence of 2-transrotational Steiner triple systems

A 2-transrotational $STS(v)$ admits an automorphism of the type $[1,1,0,\dots,0,2,0,\dots,0]$. We will construct these on the set $\{\infty,a,b\}\bigcup Z_N\times\{1,2\}$ where the automorphism is $\pi_2=(\infty)\,(a,b)\,(0_1,1_1,\dots,(N-1)_1)\,(0_2,1_2,\dots,(N-1)_2)$. We will deal with three types of differences. A *type* 1 *difference* is associated with a pair from $Z_N\times\{1\}$ and a *type* 2 *difference* is associated with a pair from $Z_N\times\{2\}$. These differences will be elements of $\{1,2,\dots,\lfloor\frac{1}{2}N\rfloor\}$. A *mixed difference*, $(j_2-i_1) \pmod N$, is associated with the pair $i_1\in Z_N\times\{1\}$ and $j_2\in Z_N\times\{2\}$ and is an element of Z_N.

Base blocks for a 2-transrotational system can be constructed according to the following rules:

(1) Base blocks may consist of elements of $Z_N \times \{1\}$ only or of $Z_N \times \{2\}$ only. A *pure* difference triple is associated with each of these except in the case of the blocks in the orbit of $(0_i, (\frac{1}{3}N)_i, (\frac{2}{3}N)_i)$ under π_2, which have the pure type i difference $\frac{1}{3}N$ associated with them.

(2) Base blocks may consist of two elements of $Z_N \times \{1\}$ and one element of $Z_N \times \{2\}$ or of one element of $Z_N \times \{1\}$ and two elements of $Z_N \times \{2\}$. A *mixed* difference triple is associated with each of these type blocks consisting of two mixed differences, d_1^M and d_2^M, and either a type 1 or a type 2 difference, d_{12}, satisfying $d_1^M + d_{12} \equiv d_2^M \pmod N$.

(3) If N is even, then the base blocks $(\infty, 0_1, (\frac{1}{2}N)_1)$ and $(\infty, 0_2, (\frac{1}{2}N)_2)$ can be used. The type 1 and type 2 differences of size $\frac{1}{2}N$ are associated with these blocks.

(4) The base block (∞, i_1, j_2) with associated mixed difference $(j_2 - i_1) \pmod N$ can be used.

(5) If N is even, then a may be put in a base block with two elements of $Z_N \times \{1\}$ if the associated type 1 difference is odd. Similarly for a base block containing a and an element of $Z_N \times \{2\}$.

(6) If N is even, then a can be in two base blocks each with an element of $Z_N \times \{1\}$ and an element of $Z_N \times \{2\}$ where either both mixed differences are even or both are odd.

(7) If $N \equiv 0 \pmod 4$ then base blocks of the forms $(a, x_1, (x + \frac{1}{2}N)_1)$, $(a, y_2, (y + \frac{1}{2}N)_2)$, and (a, u_1, v_2) where $u \not\equiv x \pmod 2$ and $v \not\equiv y \pmod 2$ can be used.

Each of the cases 5, 6, and 7 give the base blocks whose orbits contain all blocks with a or b as elements except the block (∞, a, b). From these rules, we derive the necessary conditions.

Lemma 3.1. *A necessary condition for the existence of a 2-transrotational $STS(v)$ is that* $v \equiv 3$ *or* $19 \pmod{24}$.

Proof. First, suppose that N is odd. Without loss of generality, there must be a base block of the form (a, x, y) where $x, y \in Z_N \times \{1, 2\}$. However, by applying π_2^N to this block, we see that the block (b, x, y) must also be in the $STS(v)$, a contradiction. So N must be even. Since $v = 2N + 3 \equiv 1$ or $3 \pmod 6$, it follows that $N \equiv 0$ or $2 \pmod 6$.

Case 1: Suppose that $N \equiv 2 \pmod{12}$, say $N = 12k + 2$. Then we have

Type	Difference set	Number of odd differences
1	$\{1, 2, \ldots, 6k+1\}$	$3k+1$
2	$\{1, 2, \ldots, 6k+1\}$	$3k+1$
Mixed	$\{0, 1, \ldots, 12k+1\}$	$6k+1$

So there is a total of $12k + 3$ odd differences. Base blocks containing ∞ must have associated type 1 or type 2 differences of size $\frac{1}{2}N$ (rule 3), which is odd. If not, then rule 4 must be the case and one mixed difference is associated with the ∞ containing base block; but then there are $N-1$ (which is odd) mixed differences which must be used in pairs by rules 2 and 6. Blocks containing a or b will have either two mixed differences of the same parity (rule 6) or an odd type 1 and an odd type 2 difference (rule 5). In either case, the odd differences are used in pairs. A difference triple consists of either three even numbers or of one even and two odds (from rules 1 and 2), again using odd differences in pairs. So there cannot be $12k+3$ odd differences and $N\equiv 2 \pmod{12}$ is not possible.

Case 2: Suppose $N\equiv 6 \pmod{12}$. Then we have

Type	Difference set	Number of odd differences
1	$\{1,2,\dots,6k+3\}$	$3k+2$
2	$\{1,2,\dots,6k+3\}$	$3k+2$
Mixed	$\{0,1,\dots,12k+5\}$	$6k+3$

There is a total of $12k+7$ odd differences and the argument of case 1 holds to give a contradiction here as well.

Therefore, it follows that if a 2-transrotational $STS(v)$, where $v=2N+3$, exists then $N\equiv 0$ or $8 \pmod{12}$ and $v\equiv 3$ or $19 \pmod{24}$. □

Theorem 3.2. *A 2-transrotational $STS(v)$ exists if and only if $v\equiv 3$ or $19 \pmod{24}$.*

Proof. Lemma 3.1 gives the necessary condition.

Case 1: If $v\equiv 3 \pmod{24}$, say $v=24s+3$, then consider the following blocks:

$$(\infty,a,b),\quad (\infty,0_1,(6s)_1),\quad (\infty,0_2,(6s)_2),$$

$$(a,0_1,(4s-1)_1),(a,0_2,(2s-1)_2),$$

$$(0_1,(3s-r)_1,(3s-1+r)_1)\quad \text{for } r=1,2,\dots,s-1,$$

$$(0_1,(5s-r)_1,(5s+r)_1)\quad \text{for } r=1,2,\dots,s-1,$$

$$(0_1,(4s)_1,(8s)_1),(0_2,(4s)_2,(8s)_2),$$

$$(0_1,(3s+1-r)_2,(3s+r)_2)\quad \text{for } r=1,2,\dots,s-1,s+1,s+2,\dots,3s,$$

$$(0_1,(9s-r)_2,(9s+r)_2)\quad \text{for } r=1,2,\dots,2s-1,2s+1,2s+2,\dots,3s-1,$$

$$(0_2,(8s)_1,(10s-1)_1),\quad (0_2,s_1,(3s)_1),\quad \text{and}\quad (0_1,0_2,(5s)_1).$$

These are the base blocks for a 2-transrotational $STS(v)$ where $v\equiv 3 \pmod{24}$ under π_2.

Case 2: If $v=19$ then $\{(\infty,a,b), (a,0_1,3_1), (a,0_2,3_2), (\infty,0_1,4_1), (\infty,0_2,4_2), (0_1,1_1,1_2), (0_1,2_1,4_2), (1_1,0_2,7_2), (3_1,0_2,6_2)\}$ is a set of base blocks under π_2 with $N=8$. If $v\equiv 19 \pmod{24}$ and $v>19$, say $v=24s+19$ with $s>0$, then consider the following blocks:

$$(\infty, a, b), \quad (\infty, 0_1, (6s+4)_1), \quad (\infty, 0_2, (6s+4)_2),$$

$$(a, 0_1, (6s+3)_1), (a, 0_2, (6s+3)_2),$$

$$(0_1, (1+2r)_1, (3s+3+r)_1) \quad \text{for } r=1,2,\ldots,s,$$

$$(0_1, (2r)_1, (5s+3+r)_1) \quad \text{for } r=1,2,\ldots,s-1 \quad (\text{omit if } s=1),$$

$$(0_1, (3s+3)_1, (5s+3)_1),$$

$$(0_1, (3s+1-r)_2, (3s+1+r)_2) \quad \text{for } r=1,2,\ldots,3s+1,$$

$$(0_1, (9s+7-r)_2, (9s+6+r)_2) \quad \text{for } r=1,2,\ldots,3s+1,$$

$$(0_2, (6s+4)_1, (6s+3)_1), \quad \text{and} \quad (0_2, (9s+7)_1, (6s+5)_1).$$

These are the base blocks for a 2-transrotational $STS(v)$ where $v \equiv 19 \pmod{24}$ under π_2.

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